NASA CR-1078

OPTIMUM MANEUVERS OF HYPERVELOCITY VEHICLES

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Prepared under Grant No. NGR 06-003-033 by
UNIVERSITY OF COLORADO
Boulder, Colo.

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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NOMENCLATURE

LATIN

A	constant defined by Eq. (87).
a	dimensionless total acceleration (Eq. (96)).
a_n	dimensionless normal acceleration (Eq. (95)).
a_t	dimensionless tangential acceleration (Eq. (95)).
a_1	constant of integration in the lift control (Eq.(56)).
a_2	second constant of integration in the lift control (Eq. (166)).
B	constant defined by Eq. (87).
C_D	drag coefficient
C_{D_0}	zero-lift drag coefficient
C_D^*	drag coefficient corresponding to maximum lift to drag ratio
C_L	lift coefficient
C_L^*	lift coefficient corresponding to maximum lift to drag ratio
C_i	constants, where i=1,2,....
D	drag force
E	= L/D, lift to drag ratio
E^*	maximum lift to drag ratio
g	acceleration of the gravity
H	Hamiltonian defined by Eq. (26)
J	cost function defined by Eq. (25)
K	induced drag factor defined by Eq. (1)
K_i	constants defined by Eq. (85)

NOMENCLATURE (cont.)

LATIN

L lift force

m $=\lambda^2_{opt}$, the square of the optimal control (Eq. (33))

n exponent of C_L in the generalized drag polar (Eq. (A.1))

p_i where $i=1,2,3$; components of the adjoint vector (Eqs. (26)-(29))

S reference area

t time

u dimensionless velocity defined by Eq. (19)

V velocity

W weight of the vehicle

x horizontal distance

y $=\lambda^2_{opt}-1$, defined by Eq. (158)

y_i where $i=0,1,2,\ldots$; functions in the power series expansion of y

z altitude

GREEK

α $=2E^*\log \frac{V_o}{V}$, velocity ratio defined by Eq. (69).

α_{min} minimum velocity ratio to reach the same altitude as the initial altitude (Eq. (113))

β constant length in the density altitude relation defined by Eqs. (19)

γ flight path angle

δ $=\alpha^2 - 4(\gamma_1 - \gamma_0)^2$, defined by Eq. (104).

ε $=p_1/p_3$, constant ratio defined by Eq. (38)

λ $=\sqrt{K/C_{D_0}}\, C_L$, lift control defined by Eq. (2)

λ_{max} maximum value of λ

NOMENCLATURE (cont.)

GREEK

λ_{opt} optimal value of λ

ξ dimensionless distance defined by Eq. (19)

η dimensionless altitude defined by Eq.(19)

Ω_i where i=0,1,2,...; functions in the power series expansion of ω

ω dimensionless atmospheric mass density defined by Eq. (19)

ρ atmospheric mass density

ρ_o reference atmospheric mass density

ϕ $=\sqrt{1 + a_1\gamma}$, defined by Eq. (82)

ϕ_i where i=1,2,3; roots of the denominator of the integrand in Eq. (81)

$\Delta\omega$ $= \omega_1 - \omega_o$, difference between the final and the initial dimensionless atmospheric mass density

SUBSCRIPTS (when used with the state variables V, u, γ, ξ, ω)

o denotes the initial condition

1 denotes the final condition

* denotes the condition at the inflexion point

s denotes the condition at the switching point

ABSTRACT

This report presents the analytical solutions of the problem of optimum maneuvering of a glide vehicle flying at hypervelocity regime. The investigation is based on the approximation of Allen and Eggers, namely along the fundamental part of a reentry or ascent trajectory the aerodynamic forces greatly exceed the components of the force of gravity in the directions tangent and normal to the flight path.

The problem consists of finding an optimal control law for the lift program such that the final velocity or the final altitude is maximized. As applications, the problem can be viewed as bringing the vehicle to the best condition for interception, or penetration, or making an evasive maneuver. If the range is not constrained, the solutions are obtained in closed forms. If the lift control is bounded, then bounded control is optimal whenever it is reached. The switching sequences for different cases are discussed and it is shown that there are at most two switchings.

When the range is also prescribed the problem consists of integrating a second order nonlinear system. Using Poincaré's series expansions technique, an approximate solution is obtained for the case of optimizing the final velocity with a slightly constrained range.

OPTIMUM MANEUVERS OF HYPERVELOCITY VEHICLES

I. INTRODUCTION

In recent years much attention has been focused on the optimum maneuvering of a glide vehicle. However, few analytical results have been obtained since most of the works that have been done in this area are concentrated on numerical analysis. In the light of the excellent work of Contensou [1], it appears that if the approximation of Allen and Eggers is used [2,3], that is, if the acceleration of the gravity is neglected, then a detailed analytical study of the problem can be achieved. Such analytical solutions may be less accurate than numerical solutions but they are useful in many considerations. They expose the main characteristics of the optimal law of control and permit a rapid comparative analysis of different trajectories. Also, in the range of the state space where the approximations used are justified, analytical solutions give adequate answer to the problem.

In this report we shall consider the motion of a vehicle flying at hypervelocity regime in a vertical plane with engine shut off at all points of the flight path. The vehicle trajectory can be controlled by an elevator, thus varying the aerodynamic forces acting on the flying object. The system has one independent variable, the time t, and four dependent variables, namely (Fig. 1)

x = horizontal distance

z = altitude

V = velocity

γ = flight path angle.

If we assume a drag function of the form $D = D(z,V,L)$ and if the lift program $L = L(t)$ is prescribed at all times, then for a given initial condition, the trajectory of the vehicle is uniquely determined.

The problem consists of finding an optimal control law of the lift and drag forces to bring the vehicle from a known initial flight condition to a terminal flight condition such that a certain final element, as the altitude, the range, or the velocity, is maximized.

As applications, the problem can be viewed as bringing the vehicle to the best condition for interception, or penetration, or making an evasive maneuver.

Since the analysis neglects the gravity force, the flight occurs in the fundamental part of a reentry or ascent trajectory in which, on the average, the aerodynamic forces greatly exceed the components of the force of gravity in the directions tangent and normal to the flight path. This is the portion of trajectory generally encountered in the examples mentioned above. The flight path involves a relatively short range and altitude, and hence can be investigated within the framework of the non-rotating, flat earth model.

To define a control parameter, we assume a parabolic drag polar for the vehicle (Fig.2). The relation between the drag and lift coefficients is thus given by

$$C_D = C_{Do} + KC_L^2 \tag{1}$$

where the zero-lift coefficient C_{Do} and the induced drag factor K are assumed independent of the Mach number and the Reynolds number for the velocity-altitude range of the maneuver. We shall use as control parameter the ratio of the induced drag to the zero-lift drag

$$\lambda^2 = \frac{KC_L^2}{C_{Do}} \tag{2}$$

Then we have for the lift and drag coefficients

$$C_L = \lambda C_L^* \tag{3}$$

$$C_D = 1/2\ C_D^* \ (1 + \lambda^2) \tag{4}$$

where C_L^* and C_D^* are the lift and drag coefficients corresponding to maximum lift to drag ratio E^*.

$$C_D^* = 2C_{Do}\ , \ C_L^* = \sqrt{\frac{C_{Do}}{K}}$$

$$E^* = \frac{1}{2\sqrt{KC_{Do}}} \tag{5}$$

We may assume that the control space is bounded, that is

$$|\lambda| \leq \lambda_{max} \tag{6}$$

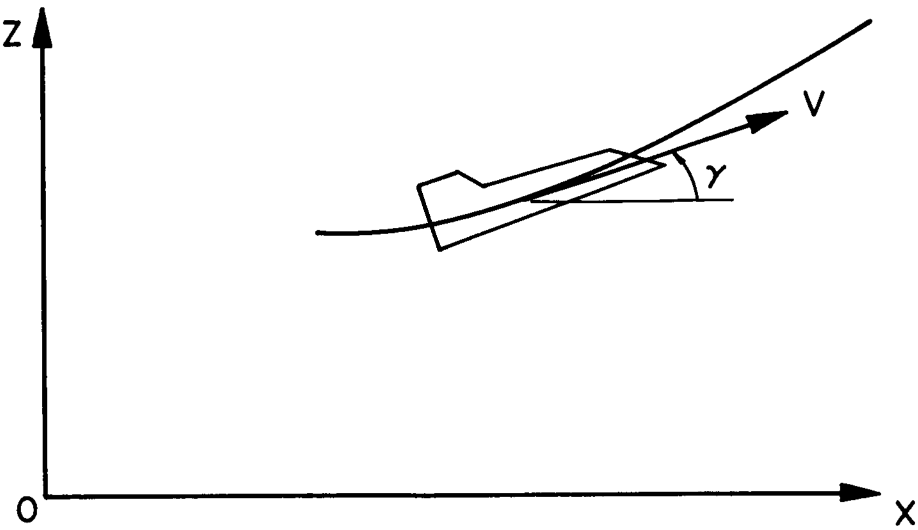

Fig. 1. GEOMETRY OF THE TRAJECTORY.

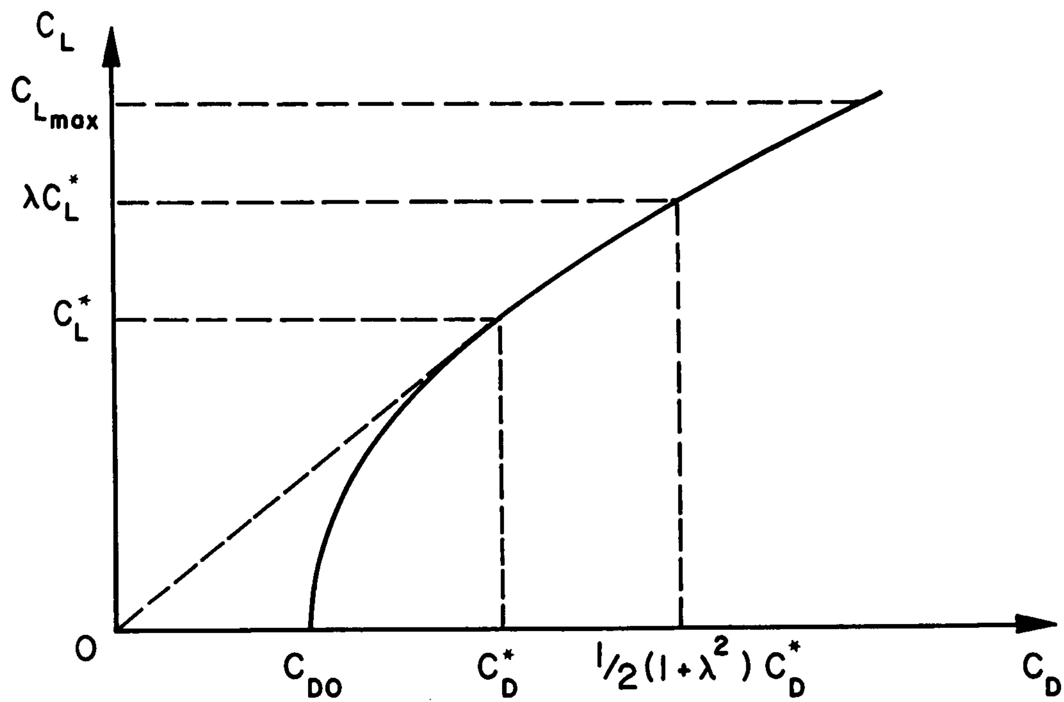

Fig. 2. PARABOLIC DRAG POLAR.

The maximum principle is a convenient variational technique to handle this case.

The plan of this report is as follows:

After the introduction in section I, the general problem is formulated as an optimal control problem in section II. In section III we give the formal solution to the problem. It will be shown that the optimal control is governed by a single nonlinear differential equation of the second order. This equation is degenerated into a first order differential equation which can be integrated in closed form if the final condition in the range is relaxed.

In sections IV, V and VI we study the degenerate case which occurs when the range is not constrained. Section IV concerns the problem of finding the trajectory which maximizes the final velocity. The problem of maximizing the final altitude is considered in section V. In section VI we analyze the case where the control is bounded. In this case bounded control may be optimal and the switching sequence is discussed.

In section VII we give approximate solution to the problem of maximizing the final velocity when the flight path angle is small and the range is slightly constrained.

The last section summarizes the principal results in this report.

II. FORMULATION OF THE PROBLEM

The motion is governed by the equations [4]

$$\frac{dx}{dt} = V \cos \gamma \tag{7}$$

$$\frac{dz}{dt} = V \sin \gamma \tag{8}$$

$$\frac{d\gamma}{dt} = \frac{gL}{VW} \tag{9}$$

$$\frac{dV}{dt} = - \frac{gD}{W} \tag{10}$$

where g is the acceleration of the gravity and W is the weight of the vehicle. If the time is eliminated, and the flight path angle is chosen as new independent variable, we have the state equations

$$\frac{dx}{d\gamma} = \frac{WV^2 \cos \gamma}{gL} \tag{11}$$

$$\frac{dz}{d\gamma} = \frac{WV^2 \sin \gamma}{gL} \tag{12}$$

$$\frac{dV}{d\gamma} = - \frac{V}{E} \tag{13}$$

where E = L/D. E and L can be explicitly expressed in terms of the control parameter.

$$E = \frac{2E^* \lambda}{1 + \lambda^2} \tag{14}$$

$$L = 1/2\, \rho S V^2 C_L^* \lambda \tag{15}$$

where S is a reference area and ρ is the atmospheric mass density. When (14) and (15) are substituted into the state equations, the result is

$$\frac{dx}{d\gamma} = \frac{2(W/S)\cos\gamma}{g\rho C_L^* \lambda} \tag{16}$$

$$\frac{dz}{d\gamma} = \frac{2(W/S)\sin\gamma}{g\rho C_L^* \lambda} \tag{17}$$

$$\frac{dV}{d\gamma} = -\frac{V(1+\lambda^2)}{2E^* \lambda} \tag{18}$$

The following dimensionless variables are now introduced.

$$\xi = \frac{x}{\beta}, \quad \eta = \frac{z}{\beta}$$

$$\omega = \frac{\rho_o g\beta C_L^*}{2(W/S)} \exp(-\eta) = \frac{g\beta C_L^*}{2(W/S)}\rho \tag{19}$$

$$u = \log\left(\frac{V}{\sqrt{\beta g}}\right)^{2E^*}$$

where the constants β and ρ_o are chosen so that the best average fit is obtained for the exponential approximation of the density variation in the altitude range of the maneuver.

In terms of the dimensionless variables, the equations of state take the simple forms

$$\frac{d\xi}{d\gamma} = \frac{\cos\gamma}{\omega\lambda} \tag{20}$$

$$\frac{d\omega}{d\gamma} = -\frac{\sin\gamma}{\lambda} \tag{21}$$

$$\frac{du}{d\gamma} = -\frac{1+\lambda^2}{\lambda} \tag{22}$$

The end conditions are:

At the initial time, $\gamma = \gamma_o$

$$\xi = \xi_o = 0 \ , \quad \omega = \omega_o \ , \quad u = u_o \tag{23}$$

At the final time, $\gamma = \gamma_1$

$$\xi = \xi_1 \ , \quad \omega = \omega_1 \ , \quad u = u_1 \tag{24}$$

It is desired to find an optimal control, λ_{opt}, to maximize a functional of the form

$$J = C_1\xi_1 + C_2\omega_1 + C_3u_1 \tag{25}$$

In this purpose we form the Hamiltonian

$$H = p_1 \frac{\cos\gamma}{\omega\lambda} - p_2 \frac{\sin\gamma}{\lambda} - p_3 \frac{1+\lambda^2}{\lambda} \tag{26}$$

where the adjoint vector (p_1, p_2, p_3) is defined by the adjoint equations

$$\frac{dp_1}{d\gamma} = -\frac{\delta H}{\delta\xi} = 0 \tag{27}$$

$$\frac{dp_2}{d\gamma} = -\frac{\delta H}{\delta\omega} = \frac{p_1\cos\gamma}{\omega^2\lambda} \tag{28}$$

$$\frac{dp_3}{d\gamma} = -\frac{\delta H}{\delta u} = 0 \tag{29}$$

with the additional end-conditions

$$p_i(\gamma_1) = - C_i \ , \qquad i = 1, 2, 3 \tag{30}$$

The solution is obtained by integrating the systems of state equations (20-22) and adjoint equations (27-29) using the end-conditions (23), (24) and (30) and a control $\lambda = \lambda_{opt}$ such that at each instant the Hamiltonian H takes on its smallest value.

III. FORMAL SOLUTION TO THE PROBLEM

We shall consider the general case where three of the state variables are prescribed at the final time and the remaining variable is to be maximized.

We successively consider the case of maximum final velocity, maximum range, and maximum final altitude. The governing equation being the same, only the end-conditions change with the problem.

First, from the adjoint equations we have immediately the integrals

$$p_1 = \text{constant}, \quad p_3 = \text{constant} \tag{31}$$

The stationary condition of the Hamiltonian implies

$$p_1 \frac{\cos\gamma}{\omega} - p_2 \sin\gamma = p_3 (1-m) \tag{32}$$

where

$$m = \lambda^2_{opt} \tag{33}$$

By differentiating (32) and using the relations (21), (28) and (31) we have

$$p_1 \frac{\sin\gamma}{\omega} + p_2 \cos\gamma = p_3 \frac{dm}{d\gamma} \tag{34}$$

Repeating the process yields

$$p_3 \frac{d^2 m}{d\gamma^2} + \left(p_2 \sin\gamma - p_1 \frac{\cos\gamma}{\omega}\right) - \frac{p_1}{\omega^2 \sqrt{m}} = 0 \tag{35}$$

Equations (32) and (34) can be solved for ω and p_2 to give

$$\frac{\varepsilon}{\omega} = \sin\gamma \frac{dm}{d\gamma} - \cos\gamma\,(m-1) \tag{36}$$

$$p_2 = p_3\left[\cos\gamma \frac{dm}{d\gamma} + \sin\gamma\,(m-1)\right] \tag{37}$$

where

$$\varepsilon = \frac{p_1}{p_3} \tag{38}$$

Finally, the substitution of Eqs. (36) and (37) into Eq. (35) yields a second order non-linear differential equation for the optimal control

$$\varepsilon\left[\frac{d^2m}{d\gamma^2} + m-1\right] = \frac{1}{\sqrt{m}}\left[\sin\gamma \frac{dm}{d\gamma} - \cos\gamma\,(m-1)\right]^2 \tag{39}$$

Concerning this equation we have the following remarks

a. Two particular solutions are

$$m = 1 \quad \text{and} \quad m = 1 + a_1 \sin\gamma \tag{40}$$

Hence we may assume a general solution of the form

$$m = 1 + a_1 \sin\gamma + \varepsilon f\,(\varepsilon,\ a_o,\ a_1,\ \gamma) \tag{41}$$

where a_o and a_1 are two constants of integration.

b. If the range is open, that is, if the final value ξ_1 is not constrained, then $p_1 = 0$. The equation can be integrated immediately to give

$$m = 1 + a_1 \sin\gamma$$

Hence for a slightly constrained range, that is, for a prescribed value ξ_1 such that it is near the value which ξ would have at the final time if it were free, we may reasonably assume that ε is small and construct approximate solution of Eq. (39) on this basis.

c. Equation (39) is of the type of the so-called singular perturbation differential equation, that is, when the perturbation vanishes we have a lower order equation. Special care must be observed in handling this type of problem since there exist boundary conditions such that for small ε the solution may not be close to the one known for $\varepsilon = 0$.

In the next three sections we will study the case where $\varepsilon = 0$ and in section VII we will derive an approximate solution for the case of maximizing the final velocity with slightly constrained range. As of now, we assume the solution (41) is known and will show how to compute the different constants of integration involved in different cases.

1. MAXIMUM FINAL VELOCITY

In this case the terminal position is prescribed, $\xi = \xi_1$, $\omega = \omega_1$ at the final time $\gamma = \gamma_1$ while the final velocity is to be maximized. The optimal control (41) contains two constants of integration a_o and a_1. Eq. (36) gives ω without integration

$$\omega = \omega\,(\gamma,\ a_o,\ a_1,\ \varepsilon) \tag{42}$$

The end-conditions in ω can be used to express a_o and a_1 in terms of ε. Next, by integrating (20) we have

$$\xi = \xi\,(\gamma,\ \varepsilon,\ C_1) \tag{43}$$

where C_1 is a new constant of integration. The end-conditions in ξ can now be used to calculate ε and C_1.

Finally (22) can be integrated to have the velocity history.

$$u = u(\gamma, C_2) \tag{44}$$

The constant of integration C_2 is obtained by the condition

$$u_o = u(\gamma_o, C_2) \tag{45}$$

while the final maximum velocity can be calculated by putting $\gamma = \gamma_1$ in Eq. (44).

2. MAXIMUM RANGE

In this case the final altitude and final velocity are prescribed, $\omega = \omega_1$, $u = u_1$ at the final time $\gamma = \gamma_1$ while the range is to be maximized.

Like the preceding case, the constant a_o and a_1 can be expressed in terms of ε by applying the end-conditions in ω to Eq. (42).

Next, by integrating (22) we have

$$u = u(\gamma, \varepsilon, C_1) \tag{46}$$

The end-conditions in u can be used to calculate ε and C_1. Finally Eq. (20) is integrated to have the range distribution

$$\xi = \xi(\gamma, C_2) \tag{47}$$

The constant of integration C_2 is obtained by the condition

$$0 = \xi(\gamma_o, C_2) \tag{48}$$

while the maximum range is obtained by putting $\gamma = \gamma_1$ in Eq. (47).

3. MAXIMUM FINAL ALTITUDE

In this case the final horizontal distance and final velocity are prescribed, $\xi = \xi_1$, $u = u_1$ at the final time $\gamma = \gamma_1$ while the final altitude is to be maximized.

First, the component p_2 of the adjoint vector is obtained without integration from Eq. (37)

$$p_2 = p_3 \, \phi(\gamma, a_o, a_1, \varepsilon) \tag{49}$$

Next, applying the initial condition to Eq. (42) and the condition $p_2(\gamma_1) = +1$ to Eq. (49) we have the equations

$$\omega_o = \omega(\gamma_o, a_o, a_1, \varepsilon) \tag{50}$$

$$1 = p_3 \phi(\gamma_1, a_o, a_1, \varepsilon) \tag{51}$$

These two equations, together with Eq. (38) are sufficient to express the constants a_o, a_1 and ε in terms of the constants p_1 and p_3. By integrating Eqs. (20) and (22) we have

$$\xi = \xi(\gamma, p_1, p_3, C_1) \tag{52}$$

$$u = u(\gamma, p_1, p_3, C_2) \tag{53}$$

The end-conditions in ξ and u give four equations to calculate the constants p_1, p_3, C_1 and C_2. Finally Eq. (21) is integrated to have the altitude distribution

$$\omega = \omega(\gamma, C_3) \tag{54}$$

The constant of integration C_3 is obtained by the initial condition

$$\omega_o = \omega\ (\gamma_o,\ C_3) \tag{55}$$

while the maximum final altitude is obtained by putting $\gamma = \gamma_1$ in Eq. (54).

In numerical applications if the optimal control, λ_{opt}, gets outside of its bound λ_{max}, then bounded control should be considered to determine the minimality of the Hamiltonian defined by (26). This case will be considered in detail in section VI.

IV. MAXIMUM FINAL VELOCITY WITH UNCONSTRAINED RANGE

When the range is not constrained, $p_1 = \varepsilon = 0$ and solution (41) is reduced to

$$\lambda^2 = 1 + a_1 \sin \gamma \tag{56}$$

This solution was obtained by Contensou [1] for the case of maximizing the final velocity of a skip trajectory when the final altitude is equal to the initial altitude. Here we shall remove that restriction.

The end-conditions are

$$\begin{aligned} &\text{At } \gamma = \gamma_o, \quad \xi = 0, \quad \omega = \omega_o, \quad u = u_o \\ &\text{At } \gamma = \gamma_1, \quad \omega = \omega_1, \quad u = \text{maximum} \end{aligned} \tag{57}$$

The constant of integration a_1 is calculated by the end-conditions in ω. We have by integrating (21)

$$\omega_1 - \omega_o = - \int_{\gamma_o}^{\gamma_1} \frac{\sin \gamma d\gamma}{\sqrt{1+a_1 \sin\gamma}} \tag{58}$$

Here we have a choice of a (+) or (-) sign in front of the radical. For a positive lift the trajectory is concave upward and by Eq. (9) γ is increasing. For a negative lift the trajectory is concave downward and the flight path angle is decreasing. If λ passes through zero and changes sign the trajectory passes through an inflexion point (Fig. 3). Expression (58) can be evaluated by the use of the elliptic integrals. Due to the heat and deceleration constraints, practical maneuvers are effectuated at small flight path angles. Then (58) is approximated by

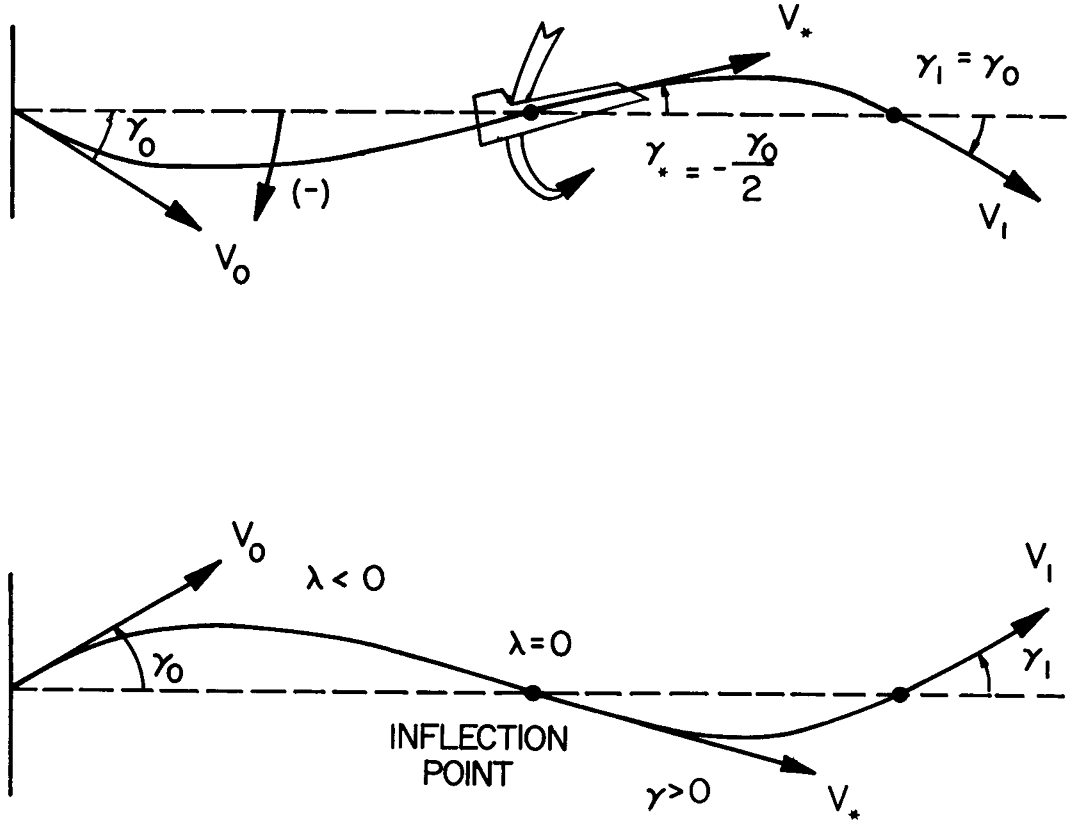

Fig. 3. OPTIMUM FLIGHT PATH WHEN $\gamma_1 = \gamma_o$, $\omega_1 = \omega_o$, SHOWING THE EFFECT OF LIFT ON CURVATURE.

$$\Delta\omega = \omega_1 - \omega_o = -\int_{\gamma_o}^{\gamma_1} \frac{\gamma d\gamma}{\sqrt{1+a_1\gamma}} \tag{59}$$

The use of γ as independent variable requires the study of the variations of γ. In the vertical plane of motion we take γ negative downward If the constant a_1 is such that λ can vanish for certain value of γ

$$\gamma_* = -\frac{1}{a_1} \tag{60}$$

then by Eq. (20), for this value of γ

$$\frac{d\gamma}{d\xi} = 0$$

and γ_* is a maximum or a minimum. The trajectory has an inflexion point. By Eq. (34)

$$\frac{d\lambda^2}{d\gamma} = \frac{p_2}{p_3}\cos\gamma \neq 0$$

Hence, λ changes sign when γ passes through γ_*. With these considerations relation (59) can be written explicitly when there is no inflexion point and in the case where we start with a positive lift

$$\frac{3a_1^2}{2}(\omega_1-\omega_o) = (2-a_1\gamma_1)\sqrt{1+a_1\gamma_1} - (2-a_1\gamma_o)\sqrt{1+a_1\gamma_o} \tag{61}$$

With an inflexion point on the trajectory, a_1 is obtained by solving

$$\frac{3a_1^2}{2}(\omega_1-\omega_o) = -(2-a_1\gamma_1)\sqrt{1+a_1\gamma_1} - (2-a_1\gamma_o)\sqrt{1+a_1\gamma_o} \tag{62}$$

In both cases we have a single quartic equation for a_1

$$\frac{81}{16}(\omega_1-\omega_o)^4 a_1^4 - \frac{9}{2}(\omega_1-\omega_o)^2(\gamma_1^3+\gamma_o^3)a_1^3$$

$$+\left[\frac{27}{2}(\omega_1-\omega_o)^2(\gamma_1^2+\gamma_o^2)+(\gamma_1^3-\gamma_o^3)^2\right]a_1^2 - 6(\gamma_1^2-\gamma_o^2)(\gamma_1^3-\gamma_o^3)a_1 \tag{63}$$

$$-36(\omega_1-\omega_o)^2+9(\gamma_1^2-\gamma_o^2)^2=0$$

In the special case where $\omega_1=\omega_o$, we have simply

$$a_1=\frac{3(\gamma_1+\gamma_o)}{\gamma_1^2+\gamma_o^2+\gamma_1\gamma_o}, \quad (\gamma_1\neq\gamma_o) \tag{64}$$

If $\omega_1=\omega_o$, $\gamma_1=\gamma_o$, a_1 is obtained by solving

$$\gamma_o^3 a_1^3 - 3\gamma_o^2 a_1^2 + 4 = 0 \tag{65}$$

or

$$a_1=-\frac{1}{\gamma_o}, \quad a_1=\frac{2}{\gamma_o} \tag{66}$$

Fig. 3 shows the trajectories in this case. For a negative γ_o, since the inflexion point occurs for positive γ, we should take $a_1=\frac{2}{\gamma_o}$. Then $\gamma_* = -\frac{\gamma_o}{2}$. The inflexion point is at the altitude $\omega_*=\omega_o$. At this point, to avoid negative acceleration the pilot can make a half roll and continue the flight with prescribed lift. The velocity ratio is

$$2E^* \log\frac{V_o}{V_1} = -4\gamma_o\sqrt{3} \tag{67}$$

For a positive γ_o, since the inflexion point occurs for negative γ we have again $a_1=\frac{2}{\gamma_o}$. The velocity ratio is the same as in the first case, and the inflexion point is at the same altitude.

When $a_1 = 0$ the maneuver is effectuated at constant angle of attack giving the maximum lift to drag ratio. No inflexion point occurs in this case, and the end conditions satisfy the relation

$$\omega_1 - \cos \gamma_1 = \omega_o - \cos \gamma_o \tag{68}$$

In the general case a_1 is obtained by solving Eq. (63). The velocity distribution is obtained by integrating Eq. (22) and we have for the case without inflexion

$$\alpha = 2E^* \log \frac{V_o}{V_1} = \frac{2}{3a_1} \left[(4+a_1 \gamma_1) \sqrt{1+a_1\gamma_1} - (4+a_1 \gamma_o) \sqrt{1+a_1 \gamma_o} \right] \tag{69}$$

Using (61) we have the alternate expression

$$\alpha = \frac{4}{a_1} \left[\sqrt{1+a_1 \gamma_1} - \sqrt{1+a_1 \gamma_o} \right] - a_1(\omega_1-\omega_o) \tag{70}$$

For the case with inflexion, we have

$$\alpha = 2E^* \log \frac{V_o}{V_1} = - \frac{2}{3a_1} \left[(4+a_1\gamma_1) \sqrt{1+a_1 \gamma_1} + (4+a_1 \gamma_o) \sqrt{1+a_1 \gamma_o} \right] \tag{71}$$

Using (62) we have the alternate expression

$$\alpha = - \frac{4}{a_1} \left[\sqrt{1+a_1 \gamma_1} + \sqrt{1+a_1 \gamma_o} \right] - a_1(\omega_1-\omega_o) \tag{72}$$

When $a_1 = 0$ we can integrate directly Eq. (22) and have the well-known expression [3]

$$\frac{V_o}{V_1} = \exp \left[\frac{(\gamma_1 - \gamma_o)}{E^*}\right] \tag{73}$$

For each value of γ_o and a given difference in the atmospheric mass density $\Delta\omega=\omega_1-\omega_o$, we can easily calculate the critical exit angle γ_1 such that the inflexion point begins to appear on the optimal trajectory by noting that such an angle and the corresponding value of a_1 satisfy both Eqs. (61) and (62). Hence, by subtracting the equations we have

$$(2-a_1\gamma_1)\sqrt{1+a_1\gamma_1} = 0 \tag{74}$$

By varying γ_1, the first inflexion point appears at the final position. Hence

$$a_1 = -\frac{1}{\gamma_1} \tag{75}$$

and from either Eq. (61) or (62) we have this critical value of γ_1 by solving

$$3(\omega_1-\omega_o) = -2\gamma_1(2\gamma_1+\gamma_o)\sqrt{\frac{(\gamma_1-\gamma_o)}{\gamma_1}} \tag{76}$$

Squaring we have a quartic equation for γ_1

$$16\gamma_1^4 - 12\gamma_o^2\gamma_1^2 - 4\gamma_o^3\gamma_1 - 9(\omega_1-\omega_o)^2 = 0 \tag{77}$$

To have the equations of the optimal trajectory, we first notice that for the altitude distribution we just have to rewrite Eqs. (61) and (62) as:

For the case without inflexion,

$$\omega(\gamma) = \frac{2}{3a_1^2}\left[(2-a_1\gamma)\sqrt{1+a_1\gamma} - (2-a_1\gamma_o)\sqrt{1+a_1\gamma_o}\right] + \omega_o \tag{78}$$

when $a_1 = 0$, ω is given by (68) with subscript 1 omitted.

For the case with inflexion point, the altitude distribution up to the inflexion point is given by Eq. (78). The altitude of the inflexion point is then

$$\omega_* = -\frac{2}{3a_1^2}(2-a_1\gamma_o)\sqrt{1+a_1\gamma_o} + \omega_o \tag{79}$$

Beyond the inflexion point the altitude distribution is given by

$$\omega(\gamma) = -\frac{2}{3a_1^2}\left[\ (2-a_1\gamma)\ \sqrt{1+a_1\gamma} + (2-a_1\gamma_o)\ \sqrt{1+a_1\gamma_o}\right] + \omega_o \quad (80)$$

The range distribution is obtained by integrating Eq. (20).

If the inflexion point is not present we have

$$\xi = \frac{3}{2a_1} \int \frac{[\phi^4 - 2\phi^2 + (1-2a_1^2)]\ d\phi}{\phi^3 - 3\phi + [(2-a_1\gamma_o)\ \sqrt{1+a_1\gamma_o} - \frac{3}{2}\ a_1^2\omega_o]} + \text{constant} \quad (81)$$

where $\phi = \sqrt{1+a_1\gamma}$ (82)

If

$$[(2-a_1\gamma_o)\ \sqrt{1+a_1\gamma_o} - \frac{3}{2}\ a_1^2\omega_o]^2 < 4 \quad (83)$$

the denominator of the integrand has three real roots, say ϕ_1, ϕ_2 and ϕ_3. Then we have explicitly

$$\xi = \frac{3}{4}\ (\gamma-\gamma_o) + \frac{3}{2a_1} \sum_{i=1}^{3} K_i \log \frac{\sqrt{1+a_1\gamma}-\phi_i}{\sqrt{1+a_1\gamma_o}-\phi_i} \quad (84)$$

where

$$K_1 = \frac{\phi_1^2 - \phi_1\ [(2-a_1\gamma_o)\ \sqrt{1+a_1\gamma_o} - \frac{3}{2}\ a_1^2\omega_o] + (1-2a_1^2)}{(\phi_1-\phi_2)\ (\phi_1-\phi_3)}$$

$$K_2 = \frac{\phi_2^2 - \phi_2\ [(2-a_1\gamma_o)\ \sqrt{1+a_1\gamma_o} - \frac{3}{2}\ a_1^2\omega_o] + (1-2a_1^2)}{(\phi_2-\phi_1)\ (\phi_2-\phi_3)} \quad (85)$$

$$K_3 = \frac{\phi_3^2 - \phi_3\ [(2-a_1\gamma_o)\ \sqrt{1+a_1\gamma_o} - \frac{3}{2}\ a_1^2\omega_o] + (1-2a_1^2)}{(\phi_3-\phi_1)\ (\phi_3-\phi_2)}$$

When $a_1 = 0$ Eq. (84) is not valid but Eq. (20) can be easily integrated by putting

$$\lambda = 1, \qquad \omega = \cos\gamma - \cos\gamma_o + \omega_o$$

This gives [4]

$$\xi = \gamma-\gamma_o - A \log\left[\frac{B + \tan\frac{\gamma}{2}}{B - \tan\frac{\gamma}{2}} \frac{B - \tan\frac{\gamma_o}{2}}{B + \tan\frac{\gamma_0}{2}}\right] \tag{86}$$

where

$$A = \frac{\omega_o - \cos\gamma_o}{\sqrt{1 - (\omega_o - \cos\gamma_o)^2}} \quad , \quad B = \sqrt{\frac{1+\omega_o - \cos\gamma_o}{1-\omega_o + \cos\gamma_o}} \tag{87}$$

In the case where the inflexion point occurs expression (84) for the range is valid up to the inflexion point, that is up to the value of $\gamma = -\frac{1}{a_1}$. The inflexion point is at a distance

$$\xi_* = -\frac{3(1+a_1\gamma_o)}{4a_1} + \frac{3}{2a_1} \sum_{i=1}^{3} K_i \log \frac{\phi_i}{\phi_i - \sqrt{1+a_1\gamma_o}} \tag{88}$$

Beyond the inflexion point (81) is replaced by

$$\xi = \frac{3}{2a_1} \int \frac{[\phi^4-2\phi^2 + (1-2a_1^2)]\, d\phi}{\phi^3-3\phi- \left[(2-a_1\gamma_o) \sqrt{1+a_1\gamma_o} - \frac{3}{2}a_1^2\omega_o\right]} + \text{constant} \tag{89}$$

If inequality (83) is satisfied the new denominator of the integrand also has three real roots, say $-\phi_1$, $-\phi_2$ and $-\phi_3$. Then we have explicitly for the range beyond the inflexion point equation (84) replaced by

$$\xi = \frac{3}{4}(\gamma-\gamma_o) + \frac{3}{2a_1} \sum_{i=1}^{3} K_i \log \frac{\phi_i + \sqrt{1+a_1\gamma}}{\phi_i - \sqrt{1+a_1\gamma_o}} \tag{90}$$

If the inequality (83) reverses, the integral (84) as well as (90) contains one arc tangent.

Fig. 4 plots the optimal trajectories in the $(-\Delta\omega, \gamma_1)$ space with a negative initial flight path angle, $\gamma_o = -10°$, and a final flight path angle $\gamma_1 \geq -\gamma_o$.

The admissible space is divided into two regions, (A) and (B), by the curve IJK. In region (A), optimum flight path is flown without inflexion. Points in region (B) are reached with an inflexion point on the optimal trajectory. The inflexion points are on the curve JK which is a quartic given by Eq. (76).

For a terminal state along the curve IJ there exist two optimal trajectories with two different lift controls, giving the same maximum final velocity. One trajectory is flown with high lift coefficient and without inflexion point. This type of trajectory easily violates condition (6) on the bounded lift control and will be analyzed in Section VI. The second trajectory is flown with lower lift coefficient and with an inflexion point.

Fig. 5 plots the optimum velocity ratio $\alpha = 2E^* \log \frac{V_o}{V_1}$ versus the final flight path angle for different $\Delta\omega$.

When $\Delta\omega \leq 0$, i.e. when the final altitude is higher than or equal to the initial altitude, there exists one value of γ_1 such that the velocity loss is minimum. This value of γ_1 is given by

$$\frac{d\alpha}{d\gamma_1} = 0$$

Or from (69)

$$\frac{da_1}{d\gamma_1}\left[\frac{8+4a_1\gamma_1-a_1^2\gamma_1^2}{\sqrt{1+a_1\gamma_1}} - \frac{8+4a_1\gamma_o-a_1^2\gamma_o^2}{\sqrt{1+a_1\gamma_o}}\right] = \frac{3a_1^2\,(2+a_1\gamma_1)}{\sqrt{1+a_1\gamma_1}} \qquad (91)$$

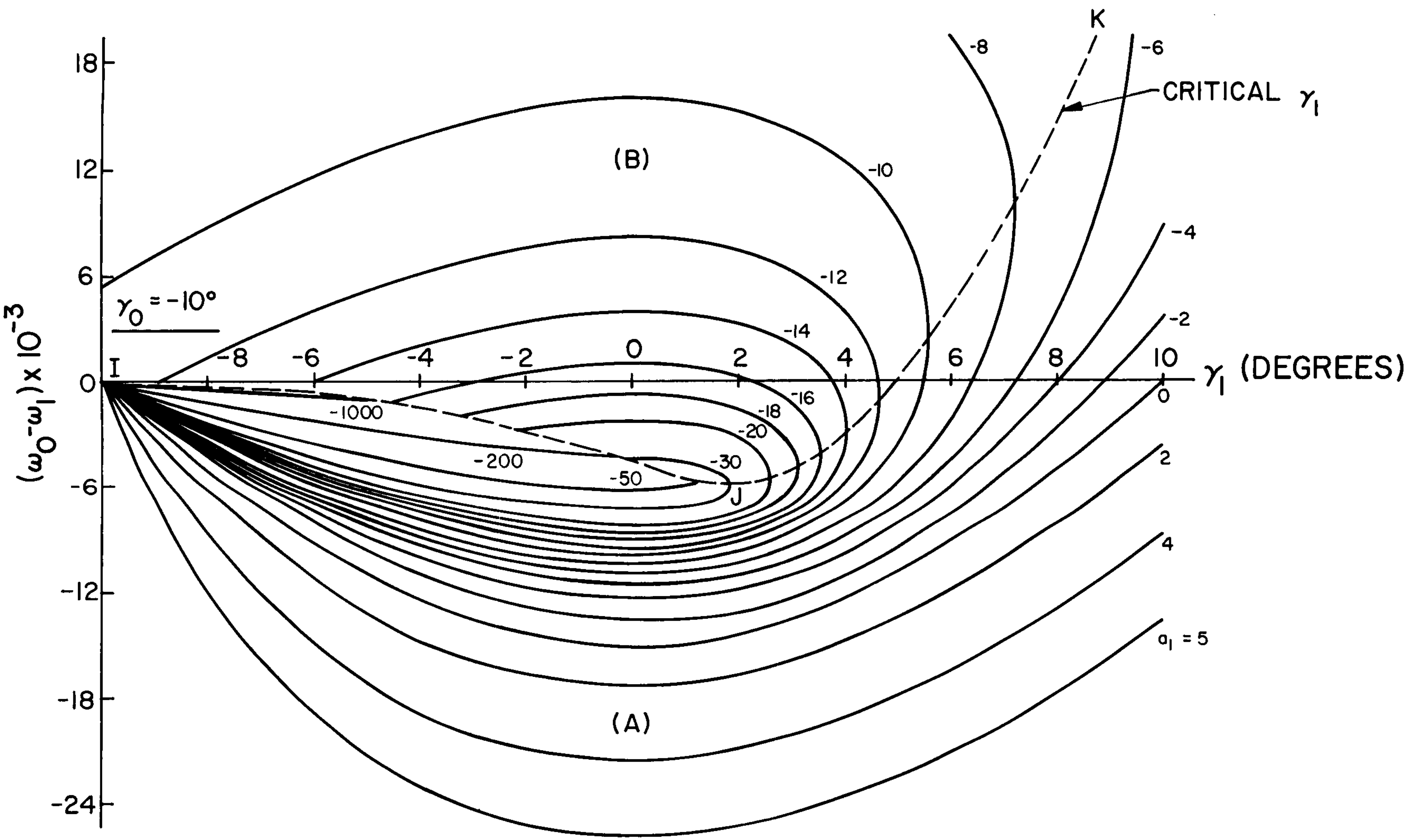

Fig. 4. OPTIMAL TRAJECTORIES FOR MAXIMUM V_1.

By differentiating (61) with respect to γ_1 we have

$$\frac{da_1}{d\gamma_1}\left[(\omega_1-\omega_o)+\frac{\gamma_1^2}{2\sqrt{1+a_1\gamma_1}}-\frac{\gamma_o^2}{2\sqrt{1+a_1\gamma_o}}\right]=-\frac{a_1\gamma_1}{2\sqrt{1+a_1\gamma_1}} \tag{92}$$

By eleminating a_1 and $\frac{da_1}{d\gamma_1}$ between the two equations above and Eq. (61) we have the equation

$$3(\omega_1-\omega_o)=-2\gamma_1\ (2\gamma_1+\gamma_o)\sqrt{\frac{(\gamma_1-\gamma_o)}{\gamma_1}} \tag{93}$$

which is identical to Eq. (76).

Hence the value of γ_1 which gives the overall minimum velocity loss for each $\Delta\omega\leq 0$ is the critical γ_1, that is the value of γ_1 such that the inflexion point appears at the final position. This critical γ_1 is obtained along the curve JK in Fig. 4.

When $\Delta\omega>0$, i.e. when the final altitude is lower than the initial altitude there exist a relative maximum velocity loss and a relative minimum velocity loss.

The maxima occur when we switch from optimum flight without inflexion to optimum flight with inflexion. For each $\Delta\omega$, the corresponding value of γ_1 is given by the curve IJ in Fig. 4. The value of γ_1 for relative minimum velocity loss is given by Eq. (93). When

$$\gamma_o\leq\gamma_1<-\frac{\gamma_o(\sqrt{3}-1)}{4}$$

$$\Delta\omega>\frac{(\sqrt{3}-1)^2\gamma_o^2}{4}\sqrt{\frac{3+2\sqrt{3}}{3}} \tag{94}$$

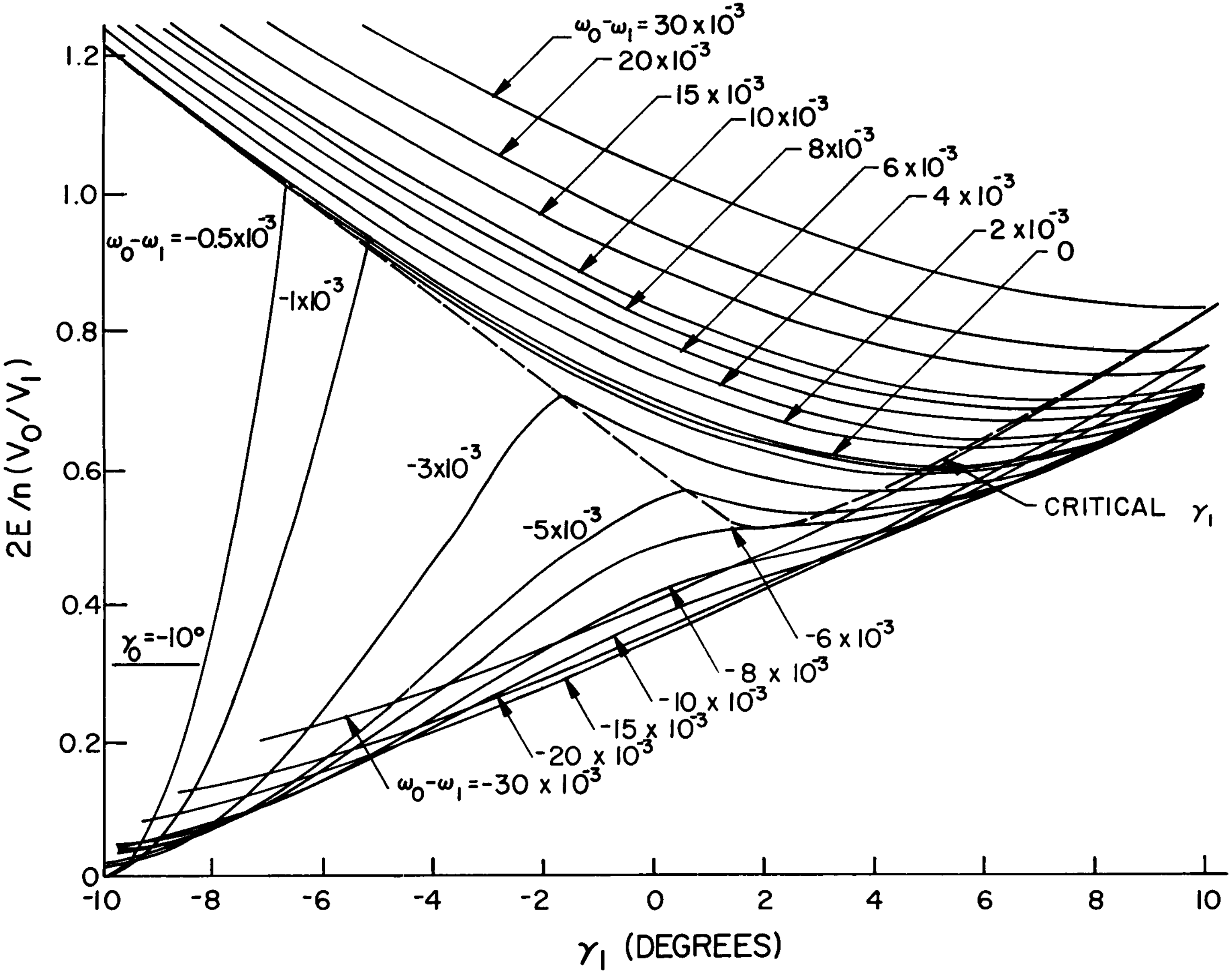

Fig. 5. OPTIMUM FINAL VELOCITY LOSS VERSUS γ_1.

relative minimum ceases to exist and all optimal trajectories are flown without inflexion.

If we also consider trajectories for which $\gamma_1<\gamma_o$, then for each $\Delta\omega>0$, there is always a critical $\gamma_1<\gamma_o$ given by (93) such that the overall velocity loss is a minimum.

We also note that for each final angle γ_1 there exists a value of $\Delta\omega$ such that the velocity loss is minimum. The trajectory is flown at maximum aerodynamic efficiency ($a_1=0$) and $\Delta\omega$ is given by Eq. (68). The line $\alpha = 2(\gamma_1-\gamma_o)$ is the envelope of the curves in Fig. 5.

The type of trajectories presented in this section is optimal in the sense that they give the minimum velocity loss.Some trajectories require high positive lift and this is not admissible if the lift control is bounded. In these cases bounded control can be optimal and discussion of this type of trajectories will be the subject of Section VI .

Also, optimal trajectories may impose high acceleration, and for completeness we will derive expressions to calculate the accelerations along the flight path.

The expressions in ω and ξ, i.e. the geometry of the optimal trajectory, do not depend on the initial velocity. Thus it is expected that if the initial velocity increases, higher accelerations will occur along the flight path. We define the dimensionless normal and tangential accelerations as

$$a_n = \frac{V\dot{\gamma}}{g}, \qquad a_t = -\frac{\dot{V}}{g} \tag{95}$$

Therefore the magnitude of the dimensionless total acceleration is

$$a = \frac{1}{g} \sqrt{(V\dot{\gamma})^2 + \dot{V}^2} \tag{96}$$

In terms of the dimensionless parameters (19) we have

$$\begin{aligned} a_n &= \omega\lambda \exp\left(\frac{u}{E^*}\right) \\ a_t &= \frac{\omega(1+\lambda^2)}{2E^*} \exp\left(\frac{u}{E^*}\right) \\ a &= \frac{\omega}{2E^*} \sqrt{(1+\lambda^2)^2 + 4E^{*2}\lambda^2} \exp\left(\frac{u}{E^*}\right) \end{aligned} \tag{97}$$

Therefore we have the relations

$$\begin{aligned} a_n &= \frac{2E^*\lambda}{1+\lambda^2} a_t \\ a &= \frac{\sqrt{(1+\lambda^2)^2 + 4E^{*2}\lambda^2}}{1+\lambda^2} a_t \end{aligned} \tag{98}$$

It can be seen that at the inflexion point the normal component of the acceleration vanishes. Expressions (97) permit the calculations of the accelerations once the optimal trajectory has been determined.

V. MAXIMUM FINAL ALTITUDE WITH UNCONSTRAINED RANGE

In this case the range is free and the optimal control is given by (56). We assume the terminal flight path angle γ_1, and the velocity loss α are prescribed and wish to find the trajectory which maximizes the final altitude.

The end-conditions are

$$\begin{aligned} &\text{At } \gamma = \gamma_o, \quad \xi = 0, \quad u = u_o, \quad \omega = \omega_o \\ &\text{At } \gamma = \gamma_1, \quad u = u_1, \quad -\omega_1 = \text{maximum} \end{aligned} \tag{99}$$

The constant of integration a_1 in the optimal control is calculated by the end-conditions in u.

$$\alpha = 2E^* \log \frac{V_o}{V_1} = \int_{\gamma_o}^{\gamma_1} \frac{2 + a_1 \sin\gamma}{\sqrt{1+a_1\sin\gamma}}\, d\gamma \tag{100}$$

The integral on the right-hand side of the equation can be calculated by the use of the elliptic integrals.

For small flight path angles and when there is no inflexion, a_1 is obtained by solving

$$\frac{3a_1}{2}\alpha = (4+a_1\gamma_1)\sqrt{1+a_1\gamma_1} - (4+a_1\gamma_o)\sqrt{1+a_1\gamma_o} \tag{101}$$

With an inflexion point on the optimal trajectory, a_1 is given by

$$\frac{3a_1}{2}\alpha = -\left[(4+a_1\gamma_1)\sqrt{1+a_1\gamma_1} + (4+a_1\gamma_o)\sqrt{1+a_1\gamma_o}\right] \tag{102}$$

In both cases we have a single quartic equation for a_1

$$(\gamma_1^3-\gamma_o^3)^2\ a_1^4 + 9[2(\gamma_1^2-\gamma_o^2)\ (\gamma_1^3-\gamma_o^3) - 1/2\alpha^2\ (\gamma_1^3+\gamma_o^3)]\ a_1^3$$

$$+ 3\left[\frac{27}{16}\alpha^4 - \frac{27}{2}\alpha^2\ (\gamma_1^2+\gamma_o^2) + 27\ (\gamma_1^2-\gamma_o^2)^2 + 16\ (\gamma_1-\gamma_o)\ (\gamma_1^3-\gamma_o^3)\right] a_1^2 \qquad (103)$$

$$- 108\ [\ \alpha^2\ (\gamma_1+\gamma_o) - 4\ (\gamma_1-\gamma_o)\ (\gamma_1^2-\gamma_o^2)]\ a_1 - 144\ [\alpha^2-4\ (\gamma_1-\gamma_o)^2] = 0$$

By putting

$$\delta = \alpha^2 - 4\ (\gamma_1-\gamma_o)^2 \qquad (104)$$

we have the alternate equation

$$(\gamma_1^3-\gamma_o^3)^2\ a_1^4 - 9\ [1/2\ (\gamma_1^3+\gamma_o^3)\ \delta - 4\gamma_1\gamma_o\ (\gamma_1+\gamma_o)\ (\gamma_1-\gamma_o)^2]\ a_1^3$$

$$+ 3\left[\frac{27}{16}\ \delta^2 - 27\gamma_1\gamma_o\delta + 16\ (\gamma_1-\gamma_o)\ (\gamma_1^3-\gamma_o^3)\right] a_1^2 \qquad (105)$$

$$- 108\ (\gamma_1+\gamma_o)\ \delta\ a_1 - 144\delta = 0$$

In this form it is readily seen that the optimum flight is effectuated at constant angle of attack giving maximum lift to drag ratio, that is, $a_1 = 0$, λ opt $= 1$, when $\delta = 0$, or equivalently when the velocity loss is such that

$$\frac{V_o}{V_1} = \exp\left(\frac{\gamma_1-\gamma_o}{E^*}\right) \qquad (106)$$

For each value of γ_o, and a given velocity loss, we can easily calculate the critical exit angle γ_1 such that the inflexion point begins to appear on the optimal trajectory by noting that such an angle and the corresponding value of a_1 satisfy both Eqs. (101) and (102). Hence, by subtracting the equations we have

$$(4+a_1\gamma_1)\sqrt{1+a_1\gamma_1} = 0 \tag{107}$$

By substituting the root $a_1 = -\frac{1}{\gamma_1}$ into either equation we have the equation for the critical γ_1

$$\frac{3}{2}\alpha = (4\gamma_1-\gamma_o)\sqrt{\frac{\gamma_1-\gamma_o}{\gamma_1}} \tag{108}$$

Squaring, we have a cubic equation for γ_1

$$16\gamma_1^3 - 24\gamma_o\gamma_1^2 + 9\left(\gamma_o^2 - \frac{\alpha^2}{4}\right)\gamma_1-\gamma_o^3 = 0 \tag{109}$$

The range and altitude distributions are calculated by the same equations as given in the preceding section. In particular the final maximum altitude is given by:

For the case without inflexion

$$\omega_1 = \frac{2}{3a_1^2}\left[(2-a_1\gamma_1)\sqrt{1+a_1\gamma_1} - (2-a_1\gamma_o)\sqrt{1+a_1\gamma_o}\right] + \omega_o \tag{110}$$

For the case with inflexion

$$\omega_1 = -\frac{2}{3a_1^2}\left[(2-a_1\gamma_1)\sqrt{1+a_1\gamma_1} + (2-a_1\gamma_o)\sqrt{1+a_1\gamma_o}\right] +\omega_o \tag{111}$$

It is obvious that, for given γ_o and γ_1, the more is the permitted velocity loss, the higher is the final altitude. If there is no gain in the final altitude, $\omega_1-\omega_o = 0$, and from Eqs. (61) and (70) we have for the case without inflexion

$$(2-a_1\gamma_1)\sqrt{1+a_1\gamma_1} = (2-a_1\gamma_o)\sqrt{1+a_1\gamma_o}$$

and (112)

$$\alpha = \frac{4}{a_1}\left[\sqrt{1+a_1\gamma_1} - \sqrt{1+a_1\gamma_o}\right]$$

By elimination of a_1 between the two equations we have

$$\alpha_{min} = 4\sqrt{\gamma_1^2+\gamma_1\gamma_o+\gamma_o^2} \tag{113}$$

This gives the minimum velocity loss to reach the same altitude as the initial altitude for the case of skip trajectory. For the case with inflexion using Eqs. (62) and (72) we have the same expression for α_{min}

Hence

$$\begin{aligned} \text{If} \quad & \alpha > \alpha_{min}, \quad \omega_1 < \omega_o \\ & \alpha = \alpha_{min}, \quad \omega_1 = \omega_o \\ & \alpha < \alpha_{min}, \quad \omega_1 > \omega_o \end{aligned} \tag{114}$$

The curve given by the Eq. (113) is a hyperbola and is plotted in dotted line in Fig. 6. This figure presents the optimal trajectories in the (α, γ_1) space with a negative initial flight path angle, $\gamma_o = -10°$ and a terminal flight path angle $\gamma_1 \geq -\gamma_o$. The admissible space is divided into two regions, (A) and (B), by the composite curve IJK.

In region (A) optimal trajectories are flown without inflexion. Points in region (B) can be reached by optimal trajectories with an inflexion point. The inflexion point occurs along the curve JK which is a cubic given by Eq. (108). For a terminal state along the curve IJ there exist two optimal trajectories, with different lift controls, giving the same maximum final altitude. One trajectory is flown with high lift and without inflexion point. This type of trajectory easily

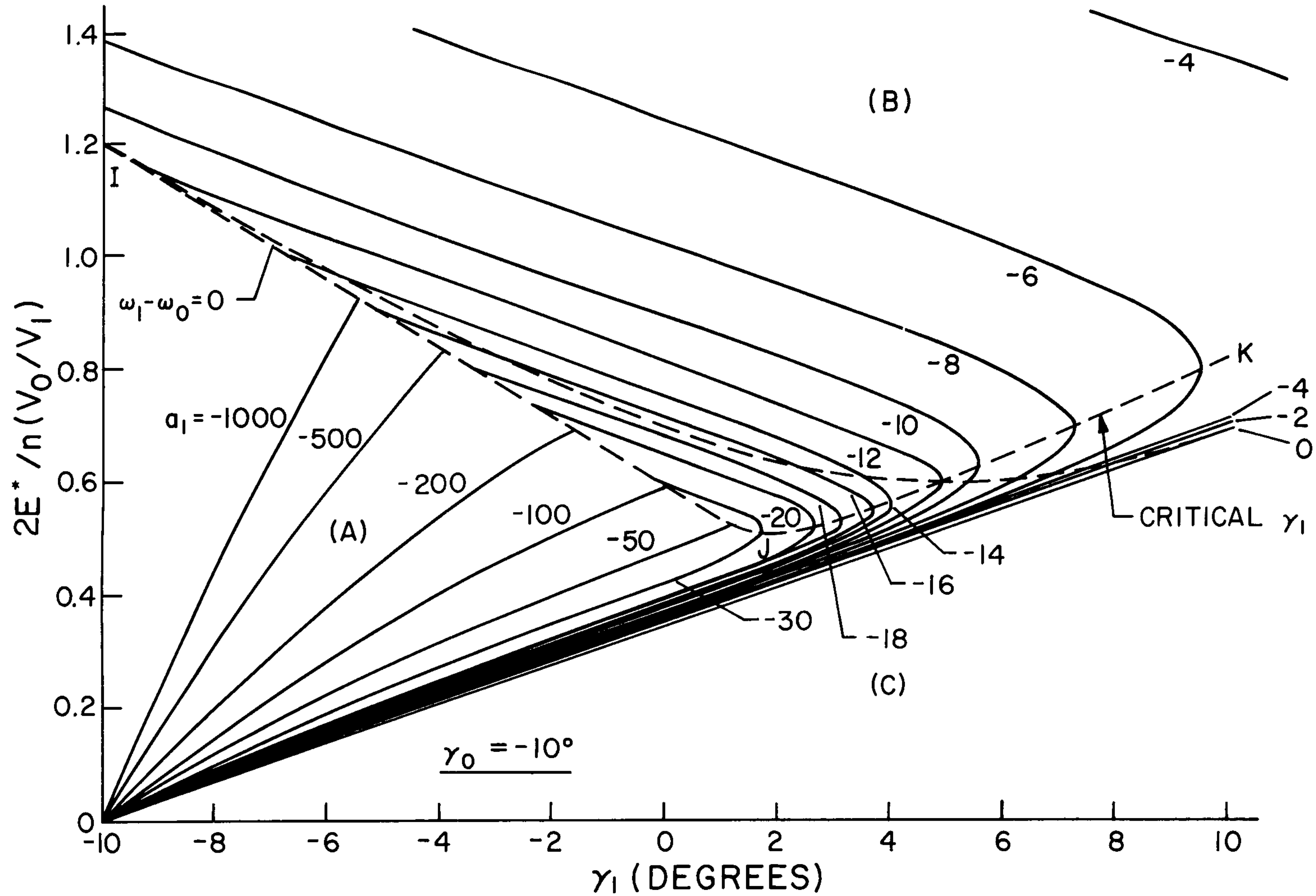

Fig. 6. OPTIMAL TRAJECTORIES FOR MAXIMUM FINAL ALTITUDE.

violates condition (6) on bounded lift control. The other trajectory requires lower lift and has an inflexion point on it.

In the graph we can notice a region (C) where no trajectory can reach. This region is below the line $a_1 = 0$, i.e. below the line

$$\alpha = 2(\gamma_1 - \gamma_o) \tag{115}$$

Below this line the prescribed velocity loss is too small for the vehicle to reach the prescribed final flight path angle. This is the problem of optimum rotation, from a given γ_o to a given γ_1, with the least velocity loss while the final position is not prescribed. Hence, it is a sub-class of the problem treated in the preceding section.

Analytically, from Eq. (32), by putting $p_1 = 0$ (free range), and $p_3 = -1$ (maximizing u_1) we have

$$\lambda^2_{opt} = 1 - p_2 \sin\gamma \tag{116}$$

Hence, $a_1 = -p_2$. If the final altitude is free $p_2 = 0$, and hence (115) gives the least velocity loss to reach a prescribed γ_1.

Fig. 7 plots the altitude gain $-\Delta\omega$ versus the terminal flight path angle γ_1, for different values of the velocity loss . When

$$\alpha < -2\sqrt{3}\gamma_o \tag{117}$$

the final altitude is always below the initial altitude. When the inequality reverses, the final maximum altitude is above the initial altitude when

$$- 1/4 \left[\sqrt{\alpha^2 - 12\gamma_o^2} + 2\gamma_o\right] < \gamma_1 < 1/4 \left[\sqrt{\alpha^2 - 12\gamma_o^2} - 2\gamma_o\right] \tag{118}$$

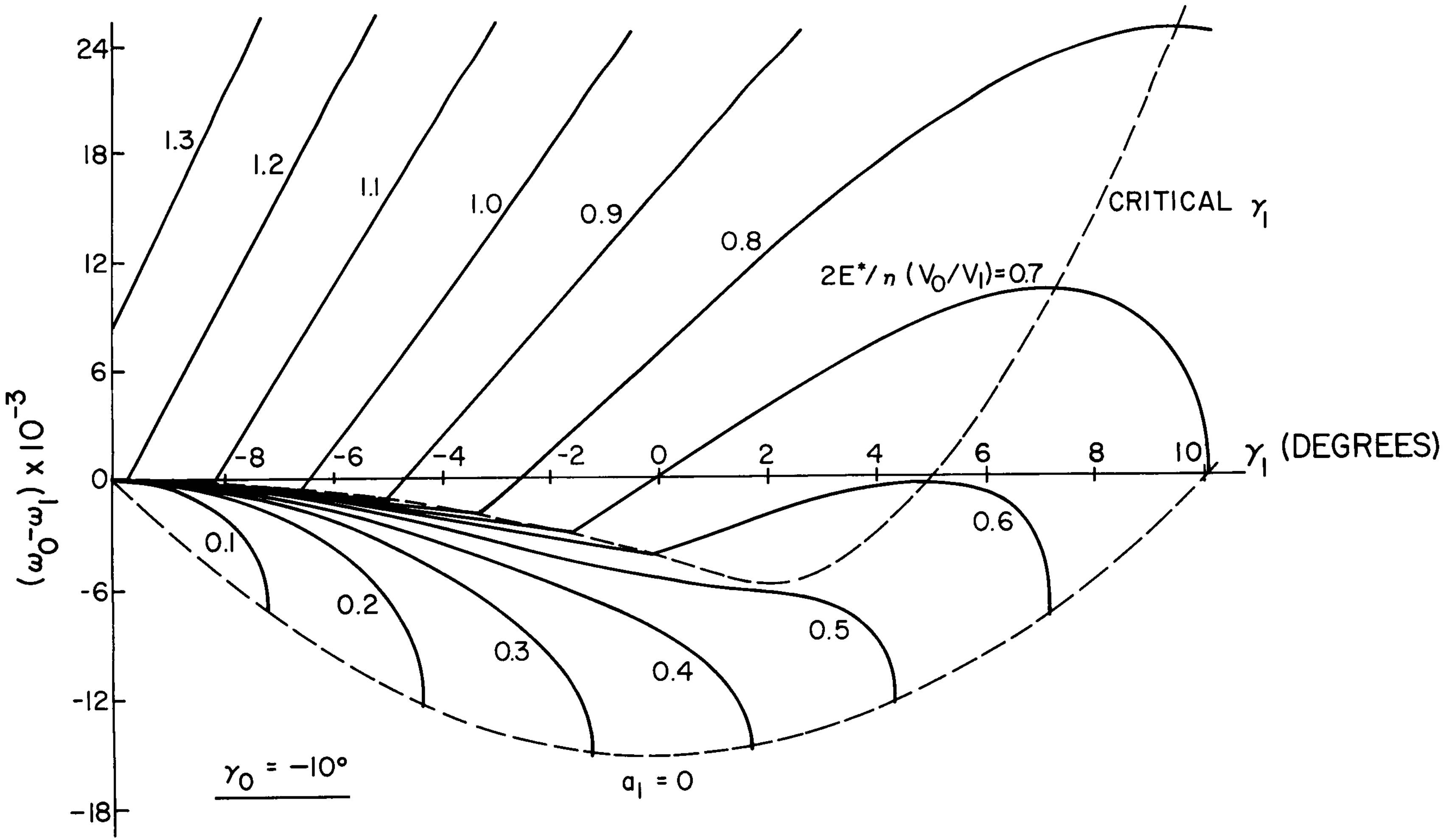

Fig. 7. OPTIMUM ALTITUDE GAIN FOR AVAILABLE VELOCITY LOSS VERSUS γ_1.

For a given velocity loss there exists a maximum final angle which can be reached given by

$$\gamma_1 = \frac{\alpha}{2} + \gamma_o \tag{119}$$

When the given velocity loss is large enough there exists a terminal flight path angle γ_1 such that the altitude gain is maximum. This angle is given by the equation

$$\frac{d\Delta\omega}{d\gamma_1} = 0$$

Using (110) we have

$$\frac{da_1}{d\gamma_1}\left[\frac{a_1^2\gamma_1^2-4a_1\gamma_1-8}{\sqrt{1+a_1\gamma_1}} - \frac{a_1^2\gamma_o^2-4a_1\gamma_o-8}{\sqrt{1+a_1\gamma_o}}\right] = \frac{3a_1^3\gamma_1}{\sqrt{1+a_1\gamma_1}} \tag{120}$$

By differentiating Eq. (101) with respect to γ_1 we have

$$\frac{da_1}{d\gamma_1}\left[\alpha - \frac{(2+a_1\gamma_1)\gamma_1}{\sqrt{1+a_1\gamma_1}} + \frac{(2+a_1\gamma_1)\gamma_o}{\sqrt{1+a_1\gamma_o}}\right] = \frac{a_1(2+a_1\gamma_1)}{\sqrt{1+a_1\gamma_1}} \tag{121}$$

By elimination of a_1 and $\frac{da_1}{d\gamma_1}$ among the two equations above and Eq. (101) we have the equation

$$\frac{3}{2}\alpha = (4\gamma_1-\gamma_o)\sqrt{\frac{\gamma_1-\gamma_o}{\gamma_1}} \tag{122}$$

which is identical to Eq. (108). Hence the γ_1 which gives the maximum altitude gain for each prescribed velocity loss is the same as the critical γ_1, that is the γ_1 such that the inflexion point is at the terminal position. When $\alpha<-2\sqrt{3}\gamma_o$ the maximum is only relative. When

$$\gamma_1 < -\frac{(\sqrt{3}-1)}{4}\gamma_o$$

$$\alpha < -2\gamma_o\sqrt{\frac{3+2\sqrt{3}}{3}} \qquad (123)$$

relative maximum altitude gain ceases to exist.

VI. OPTIMAL TRAJECTORIES WITH BOUNDED LIFT CONTROL

A simple check in Figures 4 and 6 shows that for certain prescribed terminal states the constant a_1 is such that λ_{opt} can become very large, and hence, if condition (6) on the bounded control is enforced, the optimal trajectory is not admissible.

In this section we shall study the case where bounded control is optimal. For definiteness we consider the case of negative initial flight path angle, and if inflexion point exists, the critical γ_* is such that

$$-\frac{\pi}{2} < \gamma_o \leq \gamma_1 \leq \gamma_* < \frac{\pi}{2} \tag{124}$$

We first consider the case of maximum final velocity and next, the case of maximum final altitude.

1. MAXIMUM FINAL VELOCITY

Since the range is free, and the final velocity is to be maximized

$$p_1 = 0, \quad p_2 = \text{constant} = -a_1, \quad p_3 = -1 \tag{125}$$

The control is

$$\lambda^2 = 1+a_1 \sin \gamma \tag{126}$$

$$\lambda^2 \leq \lambda^2_{max} \tag{127}$$

Expression for the Hamiltonian (26) takes the form

$$H = \frac{\lambda^2+1+a_1 \sin\gamma}{\lambda} \tag{128}$$

Initially λ is positive, and if an inflexion point exists λ changes its sign beyond the inflexion point. Since beyond the inflexion point the variable γ decreases, we should change the sign of the right-hand side of (128). Hence, expression (128) is always valid if we always take the positive sign for λ.

Fig. 8 shows the variation of H with respect to λ. The curve is a branch of hyperbola in the (H,λ) space with an absolute minimum at $\lambda = \sqrt{1+a_1 \sin\gamma}$. It is clear that if $\sqrt{1+a_1 \sin\gamma} > \lambda_{max}$, then λ_{max} is the control satisfying (127) while giving the minimum of H. Hence, λ_{max} is the optimal lift control for this case.

Fig. 9 shows the variation of λ^2 given by (126) with respect to γ for $a_1>0$. The curve intersects the line $\lambda^2=\lambda^2_{max}$ at most at one point between γ_o and γ_1. There is no inflexion point in this case. Hence, if bounded control is encountered the sequence is

$$\text{Variable control} \rightarrow \text{Bounded control} \tag{129}$$

To calculate the constant a_1 for the variable control and the value γ_s of the flight path angle at which we switch optimal control we first notice that γ_s is such that

$$\lambda^2_{max} = 1+a_1 \sin\gamma_s \tag{130}$$

Next, if we integrate Eq. (21) from γ_o to γ_s with variable control (126) we have an equation of the form

$$\omega_s-\omega_o = f_1\ (\gamma_o,\ \gamma_s,\ a_1) \tag{131}$$

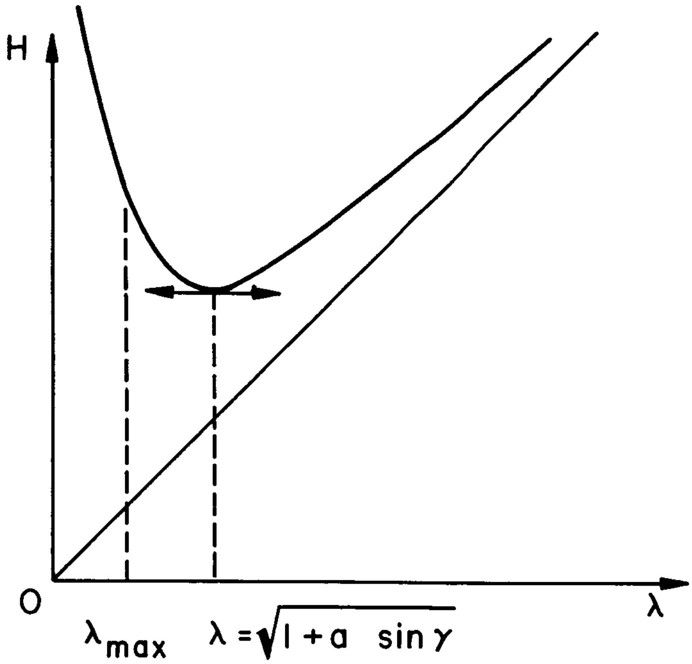

Fig. 8. VARIATION OF THE HAMILTONIAN WITH RESPECT TO λ.

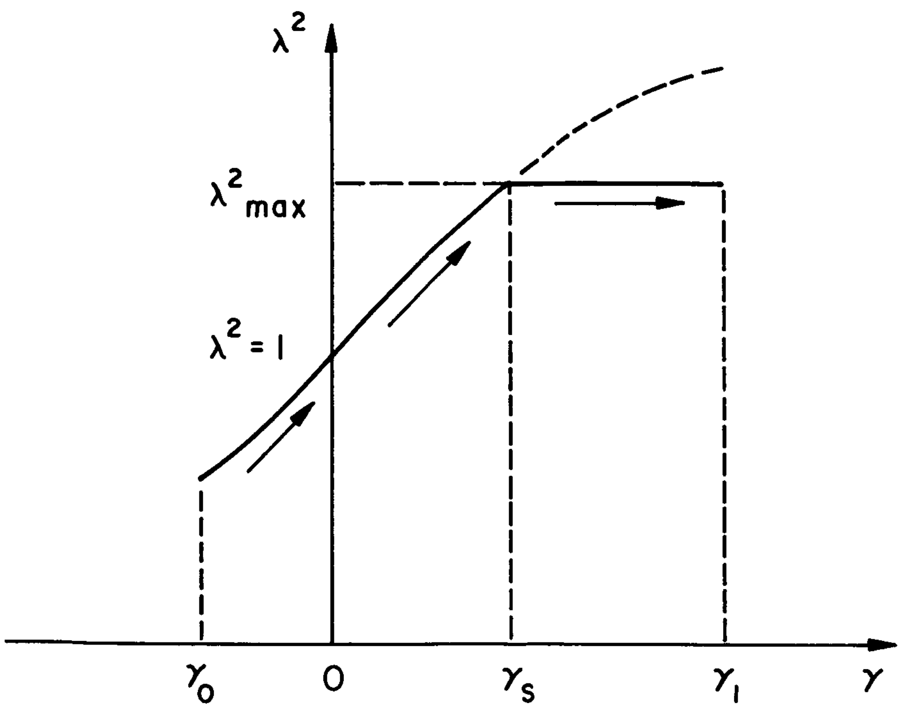

Fig. 9. VARIATION OF λ^2_{opt} WITH RESPECT TO γ (a_1>0).

where ω_s is the altitude corresponding to γ_s, i.e. the switching altitude.

Finally, if (21) is integrated from γ_s to γ_1 with $\lambda = \lambda_{max}$ we have

$$\omega_1 - \omega_s = \frac{1}{\lambda_{max}} (\cos\gamma_1 - \cos\gamma_s) \qquad (132)$$

The Eqs. (130), (131) and (132) permit the calculation of a_1, and of the switching angle γ_s and the switching altitude ω_s.

If small angle approximation is used, the equations are replaced by

$$\lambda^2_{max} = 1 + a_1\gamma_s$$

$$\omega_s - \omega_o = \frac{2}{3a_1^2}\left[(2-a_1\gamma_s)\sqrt{1+a_1\gamma_s} - (2-a_1\gamma_o)\sqrt{1+a_1\gamma_o}\right] \qquad (133)$$

$$\omega_1 - \omega_s = \frac{1}{2\lambda_{max}} (\gamma_s^2 - \gamma_1^2)$$

It can be seen that the solution to the system is given by a quartic equation in a_1.

Fig. 10 shows the variation of λ^2, given by (126), with respect to γ for $a_1<0$. The curve intersects the line $\lambda^2=\lambda^2_{max}$ at most at one point. Inflexion point may exist in this case since λ may vanish. If $\gamma_1>\gamma_s$, then the sequence for the control is

Bounded control → Variable control (134)

Variable control may be flown with or without inflexion point. To calculate the constant a_1 for the variable control we first have Eq. (130).

By integrating Eq. (21) from γ_o to γ_s with $\lambda=\lambda_{max}$ we have

$$\omega_s - \omega_o = \frac{1}{\lambda_{max}} (\cos\gamma_s - \cos\gamma_o) \qquad (135)$$

If the last portion of the trajectory is flown without inflexion point we have by integrating Eq. (21) from γ_s to γ_1 with variable control (126)

$$\omega_1-\omega_s = f_2\,(\gamma_1, \gamma_s, a_1) \tag{136}$$

The Eqs. (130), (135), and (136) permit the calculation of a_1, and of the switching angle γ_s and the switching altitude ω_s.

If the last portion of the trajectory is flown with an inflexion point, by observing the change of the sign of λ when γ passes through the critical γ_*, we can integrate Eq. (21) and replace (136) by

$$\omega_1-\omega_s = f_3\,(\gamma_1, \gamma_s, a_1) \tag{137}$$

If small angle approximation is used, we have:

For the case without inflexion

$$\begin{aligned} \lambda_{max}^2 &= 1+a_1\gamma_s \\ \omega_s-\omega_o &= \frac{1}{2\lambda_{max}}\,(\gamma_o^2-\gamma_s^2) \\ \omega_1-\omega_s &= \frac{2}{3a_1^2}\left[(2-a_1\gamma_1)\ \sqrt{1+a_1\gamma_1} - (2-a_1\gamma_s)\ \sqrt{1+a_1\gamma_s}\,\right] \end{aligned} \tag{138}$$

For the case with inflexion point the last equation is to be replaced by

$$\omega_1-\omega_s =-\frac{2}{3a_1^2}\left[(2-a_1\gamma_1)\ \sqrt{1+a_1\gamma_1} + (2-a_1\gamma_s)\ \sqrt{1+a_1\gamma_s}\,\right] \tag{139}$$

It can be seen that in either case, the resulting equation of the system is a quartic equation in a_1.

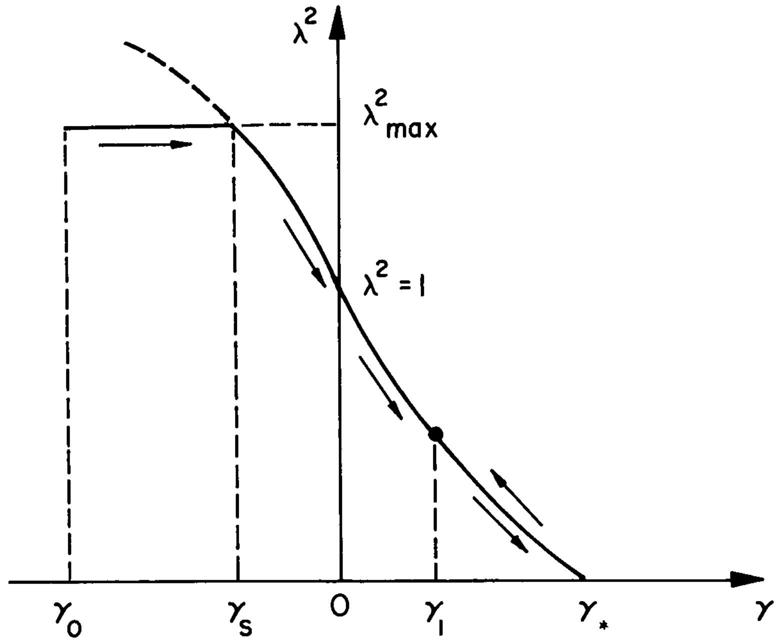

Fig. 10. VARIATION OF λ^2_{opt} WITH RESPECT TO γ (a_1<0).

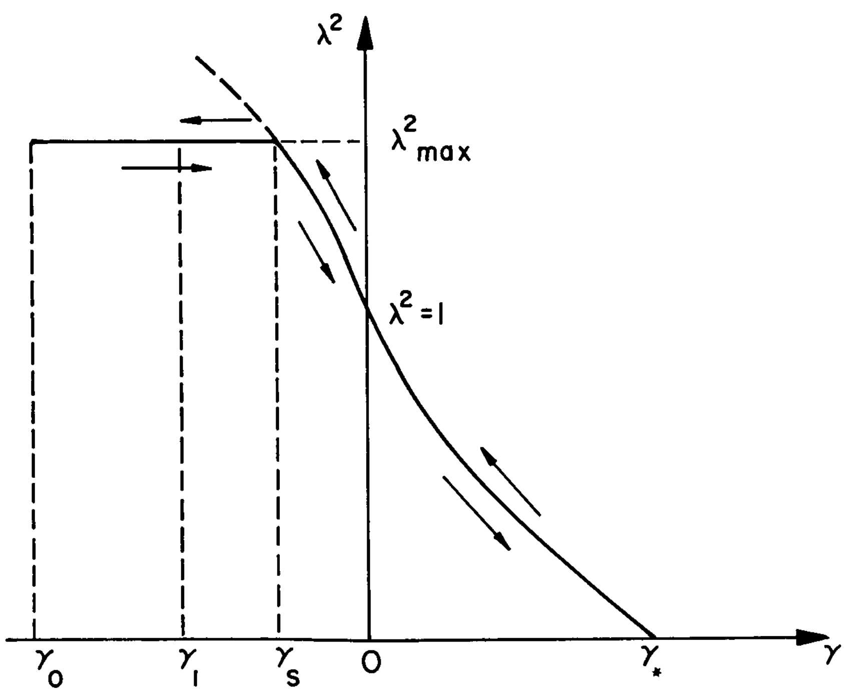

Fig. 11. CASE OF TWO SWITCHINGS.

It is possible that, for the case where $a_1<0$, the end-conditions and the value of λ_{max} are such that $\gamma_1<\gamma_s$ (Fig. 11). For this case it is necessary that an inflexion point exists as it can be seen in the figure. For this case the sequence for the control is

Bounded control → Variable control (with inflexion) → Bounded control (140)

To calculate the optimal elements involved we first have Eq. (130). Then by integrating (21) from γ_o to γ_s with $\lambda=\lambda_{max}$ we have

$$\omega_{s1}-\omega_o = \frac{1}{\lambda_{max}}(\cos\gamma_s-\cos\gamma_o) \tag{141}$$

where ω_{s1} is the first switching altitude (Fig. 12). Next, by integrating (21) from γ_s to γ_* with positive variable lift, and then from γ_* to γ_s with negative variable lift we have an expression of the form

$$\omega_{s2}-\omega_{s1} = f_4\ (\gamma_s,\ a_1) \tag{142}$$

where ω_{s2} is the second switching altitude. Finally the equation for ω is integrated from γ_s to γ_1 with $\lambda=-\lambda_{max}$ to give

$$\omega_1-\omega_{s2} = \frac{1}{\lambda_{max}}(\cos\gamma_s-\cos\gamma_1) \tag{143}$$

The system of Eqs. (130), (141), (142) and (143) gives a_1, γ_s, ω_{s1} and ω_{s2}.

If small angle approximation is used we have the system

$$\begin{aligned}
\lambda^2_{max} &= 1+a_1\gamma_s \\
\omega_{s1}-\omega_o &= \frac{1}{2\lambda_{max}}(\gamma_o^2-\gamma_s^2) \\
\omega_{s2}-\omega_{s1} &= -\frac{4(2-a_1\gamma_s)}{3a_1^2}\sqrt{1+a_1\gamma_s} \\
\omega_1-\omega_{s2} &= \frac{1}{2\lambda_{max}}(\gamma_1^2-\gamma_s^2)
\end{aligned} \tag{144}$$

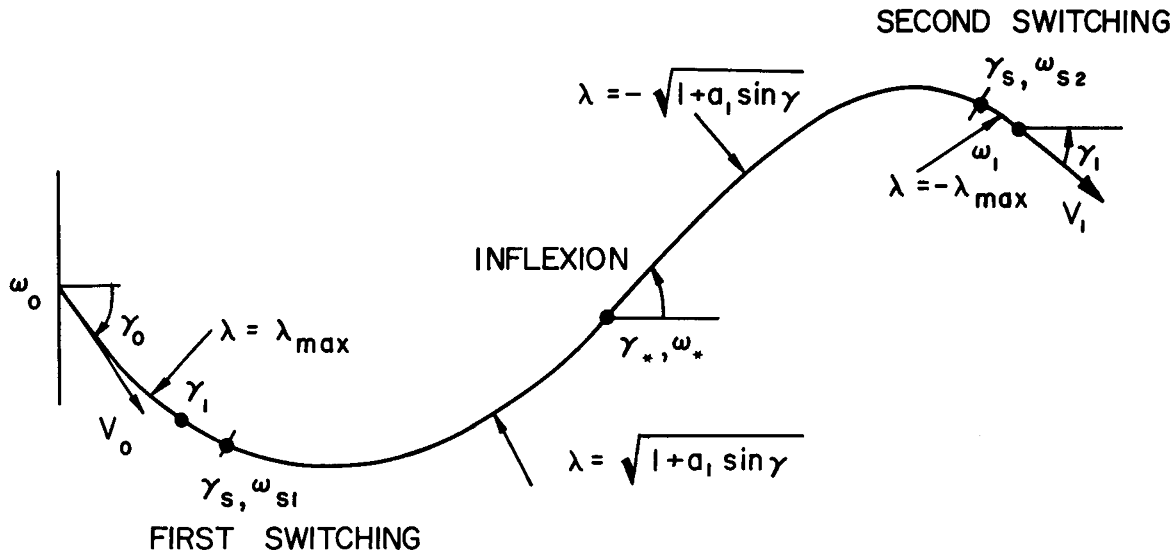

Fig. 12. OPTIMAL FLIGHT PATH WITH TWO SWITCHINGS.

By solving the system we have

$$a_1 = - \sqrt{\frac{2(-\lambda_{max}^4+6\lambda_{max}^2+3)}{3\left[\gamma_o^2+\gamma_1^2-2\lambda_{max}\ (\omega_1-\omega_o)\right]}} \tag{145}$$

and from this value of a_1 we can easily calculate γ_s, ω_{s1} and ω_{s2}. If

$$\lambda_{max}^2 < 3+2\sqrt{3} \tag{146}$$

we must have

$$\omega_1-\omega_o < \frac{\gamma_o^2+\gamma_1^2}{2\lambda_{max}} \tag{147}$$

Otherwise both inequalities reverse for real value of a_1.

Since bounded control interferes only with high lift trajectories, it is possible for the same end-condition to find a lower lift variety giving better saving in the velocity loss as explained before. Hence for the case of bounded control we must compare the trajectory with alternate lower lift trajectory to search the true optimal trajectory.

2. MAXIMUM FINAL ALTITUDE

Since the range is free, and the final altitude is to be maximized, we have

$$p_1=0, \qquad p_2=1, \qquad p_3=\text{constant}=\frac{1}{a_1} \tag{148}$$

Expression for the Hamiltonian (26) takes the form

$$H = -\frac{1}{a_1}\frac{\lambda^2+1+a_1\sin\gamma}{\lambda} \tag{149}$$

For positive lift, if a_1 is positive the Hamiltonian is an absolute minimum for $\lambda=0$ and the trajectory cannot lead to maximum final altitude. Hence, we have only negative values for a_1 as it can be seen in Fig. 6. Thus, in this case only Figs. (10) and (11) apply.

The arguments are the same as in the case of maximum final velocity. The integration of Eq. (21) is to be replaced by the integration of Eq. (22). We give here the relations to calculate the optimal elements for the case of small flight path angles.

In the case of one switching and without inflexion we have the system

$$\lambda_{max}^2 = 1+a_1\gamma_s$$

$$2E^*\log \frac{V_o}{V_s} = \frac{1+\lambda_{max}^2}{\lambda_{max}} (\gamma_s-\gamma_o) \qquad (150)$$

$$2E^*\log \frac{V_s}{V_1} = \frac{2}{3a_1}\left[(4+a_1\gamma_1)\sqrt{1+a_1\gamma_1} - (4+a_1\gamma_s)\sqrt{1+a_1\gamma_s}\right]$$

where V_s is the velocity at the switching point. When there is an inflexion on the variable lift portion of the trajectory, the last equation is to be replaced by

$$2E^*\log \frac{V_s}{V_1} = -\frac{2}{3a_1}\left[(4+a_1\gamma_1)\sqrt{1+a_1\gamma_1} + (4+a_1\gamma_s)\sqrt{1+a_1\gamma_s}\right] \qquad (151)$$

In both cases the resulting equation is a cubic equation in a_1.

In the case of two switchings, the variable lift portion of the trajectory is flown with an inflexion and the solution is obtained by solving the system

$$\lambda_{max}^2 = 1+a_1\gamma_s$$

$$2E^*\log\frac{V_o}{V_{s1}} = \frac{1+\lambda_{max}^2}{\lambda_{max}}(\gamma_s-\gamma_o)$$

$$2E^*\log\frac{V_{s1}}{V_{s2}} = -\frac{4(4+a_1\gamma_s)}{3a_1}\sqrt{1+a_1\gamma_s} \tag{152}$$

$$2E^*\log\frac{V_{s2}}{V_1} = \frac{1+\lambda_{max}^2}{\lambda_{max}}(\gamma_s-\gamma_1)$$

By solving the system we have

$$a_1 = \frac{2(\lambda_{max}^4 - 6\lambda_{max}^2 - 3)}{3\left[(1+\lambda_{max}^2)(\gamma_o+\gamma_1)+\lambda_{max}\alpha\right]} \tag{153}$$

For negative value of a_1, if

$$\lambda_{max}^2 < 3+2\sqrt{3} \tag{154}$$

we must have

$$\alpha > -\frac{(1+\lambda_{max}^2)(\gamma_o+\gamma_1)}{\lambda_{max}} \tag{155}$$

VII. APPROXIMATE SOLUTION FOR SLIGHTLY CONSTRAINED RANGE

The singular perturbation differential equation (39) can only be handled with specific boundary conditions because of the different orders of magnitude of the elements involved. To take an example, in this section we shall consider the problem of maximizing the final velocity of a hypervelocity interceptor. The vehicle is coming from an initial position, $\xi_o=0$, ω_o, at an angle γ_o, with an initial velocity u_o. It is proposed to bring the interceptor along a flight path to a new position ξ_1, ω_1, at an angle γ_1 such that the final velocity u_1 is a maximum. If the prescribed value ξ_1 is such that it is near the value which ξ would have at the final time if the range is free, then it is reasonable to expect that the parameter ε would be small. Furthermore, we shall take an altitude range such that the quantity ε/ω is small.

Instead of the governing equation (39) we shall consider the system

$$\sin\gamma y' - \cos\gamma y = \frac{\varepsilon}{\omega} \tag{156}$$

$$\omega' = -\frac{\sin\gamma}{\sqrt{1+y}} \tag{157}$$

where

$$1+y = m = \lambda_{opt}^2 \tag{158}$$

and the prime denotes the differentiation with respect to γ. We seek solutions to the system of the form

$$\begin{aligned} y &= y_o + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots \\ \omega &= \Omega_o + \varepsilon\Omega_1 + \varepsilon^2\Omega_2 + \ldots \end{aligned} \tag{159}$$

where y_i and Ω_i are functions of γ. It has been shown by Poincaré that the series converge if ε is reasonably small [5].

Upon substituting the series into the system, and equating like powers in ε we have

$$\begin{aligned}
&\sin\gamma y_o' - \cos\gamma y_o = 0 \\
&\sin\gamma y_1' - \cos\gamma y_1 = \frac{1}{\Omega_o} \\
&\sin\gamma y_2' - \cos\gamma y_2 = -\frac{\Omega_1}{\Omega_o^2} \\
&\ldots\ldots\ldots
\end{aligned} \tag{160}$$

and

$$\begin{aligned}
&\Omega_o' = -\frac{\sin\gamma}{\sqrt{1+y_o}} \\
&\Omega_1' = \frac{\sin\gamma y_1}{2(1+y_o)^{3/2}} \\
&\Omega_2' = -\frac{\sin\gamma}{2}\left[\frac{3y_1^2}{4(1+y_o)^{5/2}} - \frac{y_2}{(1+y_o)^{3/2}}\right] \\
&\ldots\ldots\ldots
\end{aligned} \tag{161}$$

By integrating the first equation (160) we have to the zeroth order

$$y_o = a_1 \sin\gamma \tag{162}$$

By substituting this expression for y_o into the first equation (161) and integrating we have

$$\Omega_o(\gamma) = -\int \frac{\sin\gamma d\gamma}{\sqrt{1+a_1 \sin\gamma}} + \text{constant} \tag{163}$$

The constants of integration are evaluated by the end-conditions in ω by assuming

$$\Omega_o(\gamma_o) = \omega_o, \qquad \Omega_o(\gamma_1) = \omega_1 \tag{164}$$

Using small angle approximation, we have for the case without inflexion

$$\Omega_o(\gamma) = \frac{2}{3a_1^2}\left[(2-a_1\gamma)\ \sqrt{1+a_1\gamma} - (2-a_1\gamma_o)\ \sqrt{1+a_1\gamma_o}\right] + \omega_o \tag{165}$$

where a_1 is given by Eq. (61).

To have a first order solution we integrate the second equation (160) using the expression above for Ω_o. This gives

$$y_1(\gamma) = \gamma\left[\int \frac{3a_1^2 d\gamma}{\gamma^2\left[2(2-a_1\gamma)\ \sqrt{1+a_1\gamma} - 2(2-a_1\gamma_o)\ \sqrt{1+a_1\gamma_o} + 3a_1^2\omega_o\right]} + a_2\right] \tag{166}$$

where a_2 is a new constant of integration. By the change of variable $\phi = \sqrt{1+a_1\gamma}$ we have a rational integral and its integration gives explicitly

$$y_1(\gamma) = \gamma\left[f(\gamma) + a_2\right] \tag{167}$$

Next by using y_1 in the second equation (161) and integrating we have

$$\Omega_1(\gamma) = \int \frac{\gamma^2 f(\gamma) d\gamma}{2(1+a_1\gamma)^{3/2}} + a_2 \int \frac{\gamma^2 d\gamma}{2(1+a_1\gamma)^{3/2}} + a_3 \tag{168}$$

By the change of variable $\phi = \sqrt{1+a_1\gamma}$ the second integral can be easily integrated. The first integral contains terms in the forms

$$\int \phi^p \log\ (\phi^2+b) d\phi$$

$$\int \phi^p \log\ (\phi+b) d\phi \qquad \text{and} \qquad \int \phi^p \arctan\ (\phi+b)\ d\phi$$

These integrals can also be expressed in terms of elementary functions.

Finally, the constants a_2 and a_3 in Eq. (168) are calculated by the end-conditions

$$\Omega_1(\gamma_o) = \Omega_1(\gamma_1) = 0 \tag{169}$$

The optimal control defined by (158) is now of the form

$$\lambda = \lambda(\gamma, \varepsilon) \tag{170}$$

The constant ε is calculated by integrating the equation (20) in ξ, following the procedure outlined in section III. To the first order in ε

$$\xi = \int \frac{(1 - \frac{\gamma^2}{2})d\gamma}{\Omega_o \sqrt{1+y_o}} - \varepsilon\left[\int \frac{(1 - \frac{\gamma^2}{2})\,\Omega_1 d\gamma}{\Omega_o^2 \sqrt{1+y_o}} + \int \frac{(1 - \frac{\gamma^2}{2})\, y_1 d\gamma}{2\Omega_o(1+y_o)^{3/2}}\right] + \text{constant} \tag{171}$$

The first integral is the same as integral (81). It represents the unperturbed range. The new constant of integration in (171) and the parameter ε are now calculated by the end-conditions in ξ. When ε is known, the velocity history is obtained by integrating Eq. (22). The problem is hence completely solved.

VIII. CONCLUSION

In this report the analytical solutions of some problems concerning the optimum maneuvering of a lifting vehicle in the hypervelocity regime have been obtained using the approximation of Allen and Eggers.

If the range is free, the optimal lift control is obtained in closed form. It is interesting to notice that, although the pertinent formulas derived in this report have been obtained with a parabolic drag polar, the problems considered are still solvable analytically with a generalized drag polar as it is shown in Appendix A.

An interesting feature in the optimal trajectory is that it may have an inflexion point at which the lift changes sign by passing through zero. The optimum cost functions (maximum final velocity or maximum final altitude as they are discussed in this report) are stationary when the inflexion is at the terminal position. This property is proved in the text using small angle approximation. But it is also true when the exact equations are considered. The proof of this property for large angles is given in Appendix B.

The study includes the case where the lift control is bounded. In this case bounded control is optimal whenever it is reached. The switching sequences for different cases are discussed. It is shown that for the case of free range there are at most two switchings. Bounded lift control is always at the two ends of the optimal trajectory.

For the general case, when the range is also prescribed the problem consists of integrating a second order nonlinear system. An approximate solution is obtained for the case where the range is slightly constrained. It is the hope of the authors that other integrable cases will be obtained by the readers who are interested in other aspects of this fascinating problem.

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APPENDIX A.

OPTIMAL CONTROL FOR GENERALIZED DRAG POLAR

Consider the equation of the generalized drag polar

$$C_D = C_{Do} + KC_L^{\,n} \tag{A.1}$$

It can be employed to represent the relation between the lift and drag coefficients in any flow regime if the zero-lift drag coefficient C_{Do}, the induced drag factor K, and the exponent n are regarded to be functions of both the Mach number and the Reynolds number. Here we shall assume that C_{Do}, K and n are constant in the velocity-altitude range considered. For thin-winged configurations operating in the hypervelocity domain, n is close to 3/2.

We define a control parameter λ by the relation

$$\lambda^n = \frac{(n-1)KC_L^{\,n}}{C_{Do}} \tag{A.2}$$

Then for each value of λ, the lift and drag coefficients are given by

$$\begin{aligned} C_L &= C_L^* \lambda \\ C_D &= \frac{1}{n} C_D^* \left[(n-1) + \lambda^n\right] \end{aligned} \tag{A.3}$$

where C_L^* and C_D^* are the lift and drag coefficients corresponding to maximum lift to drag ratio E*.

$$C_L^* = \left(\frac{C_{Do}}{(n-1)K}\right)^{\frac{1}{n}}, \qquad C_D^* = \frac{n}{n-1} C_{Do}$$

$$E^* = \frac{1}{n}\left(\frac{n-1}{C_{Do}}\right)^{\frac{n-1}{n}} \left(\frac{1}{K}\right)^{\frac{1}{n}} \tag{A.4}$$

We notice that $\lambda=1$ corresponds to maximum lift to drag ratio.

Also we may have

$$|\lambda| \leq \lambda_{max} \tag{A.5}$$

The dimensionless variables are defined as

$$\xi = \frac{X}{\beta}, \qquad \eta = \frac{z}{\beta}$$

$$\omega = \frac{\rho_o g\beta C_L^*}{2(W/S)} \exp(-\eta) = \frac{g\beta C_L^*}{2(W/S)} \rho \tag{A.6}$$

$$u = nE^*\log\left(\frac{V}{\sqrt{\beta g}}\right)$$

Then, the state equations (20), (21) and (22) are replaced by

$$\frac{d\xi}{d\gamma} = \frac{\cos\gamma}{\omega\lambda}$$

$$\frac{d\omega}{d\gamma} = -\frac{\sin\gamma}{\lambda} \tag{A.7}$$

$$\frac{du}{d\gamma} = -\frac{(n-1)+\lambda^n}{\lambda}$$

It can be seen that for the case where the range is free, the stationary values of the Hamiltonian correspond to

$$\lambda_{opt} = \sqrt[n]{1+a_1\sin\gamma} \tag{A.8}$$

This value of λ and the value λ_{max} constitute the set of possible optimal controls for positive lift.

APPENDIX B.

INFLEXION POINT IN THE CASE OF LARGE ANGLE

We shall consider the case of free range - maximum final velocity with parabolic drag polar.

The constant of integration a_1 in the optimal control is calculated by

$$\Delta\omega = \omega_1 - \omega_o = - \int_{\gamma_o}^{\gamma_1} \frac{\sin\gamma d\gamma}{\sqrt{1+a_1\sin\gamma}} \tag{B.1}$$

The critical value of γ_1 has been defined as the one such that the inflexion point is at the terminal position.

$$\sin\gamma_1 = - \frac{1}{a_1} \tag{B.2}$$

Hence, the critical γ_1 is obtained by solving

$$\Delta\omega = - \int_{\gamma_o}^{\gamma_1} \frac{\sin \gamma d\gamma}{\sqrt{1 - (\sin\gamma/\sin\gamma_1)}} \tag{B.3}$$

In the case of small angle, we have the approximate equation (76).

For the case without inflexion, the velocity ratio is given by

$$\alpha = 2E^*\log \frac{V_o}{V_1} = \int_{\gamma_o}^{\gamma_1} \frac{2+a_1 \sin\gamma}{\sqrt{1+a_1 \sin\gamma}} d\gamma \tag{B.4}$$

or, using (B.1)

$$\alpha = 2\int_{\gamma_o}^{\gamma_1} \frac{d\gamma}{\sqrt{1+a_1 \sin\gamma}} - a_1 \Delta\omega \tag{B.5}$$

For minimum velocity loss

$$\frac{d\alpha}{d\gamma_1} = -\int_{\gamma_o}^{\gamma_1} \frac{\sin\gamma}{(1+a_1\sin\gamma)^{3/2}} \frac{da_1}{d\gamma_1} d\gamma + \frac{2}{\sqrt{1+a_1\sin\gamma_1}} - \Delta\omega \frac{da_1}{d\gamma_1} = 0 \tag{B.6}$$

By differentiating (B.1) with respect to γ_1

$$\int_{\gamma_o}^{\gamma_1} \frac{\sin^2\gamma}{2(1+a_1\sin\gamma)^{3/2}} \frac{da_1}{d\gamma_1} d\gamma - \frac{\sin\gamma_1}{\sqrt{1+a_1\sin\gamma_1}} = 0 \tag{B.7}$$

Therefore

$$\frac{da_1}{d\gamma_1}\left[\Delta\omega + \int_{\gamma_o}^{\gamma_1} \frac{\sin\gamma d\gamma}{(1+a_1\sin\gamma)^{3/2}}\right] = \frac{2}{\sqrt{1+a_1\sin\gamma_1}}$$

$$\frac{da_1}{d\gamma_1}\int_{\gamma_o}^{\gamma_1} \frac{\sin^2\gamma d\gamma}{2(1+a_1\sin\gamma)^{3/2}} = \frac{\sin\gamma_1}{\sqrt{1+a_1\sin\gamma_1}} \tag{B.8}$$

By eliminating $\Delta\omega$ and $\frac{da_1}{d\gamma_1}$ among the two equations above and Eq. (B.1) we have

$$(1+a_1\sin\gamma_1)\int_{\gamma_o}^{\gamma_1} \frac{\sin^2\gamma d\gamma}{(1+a_1\sin\gamma)^{3/2}} = 0 \tag{B.9}$$

Hence, we have

$$1+a_1\sin\gamma_1 = 0 \tag{B.10}$$

This shows that the critical values of γ_1, given by Eq. (B.3) are the values of γ_1 corresponding to the stationary values of α.

Following the same type of derivation we can show that the critical γ_1 in the case of maximum final altitude is given by

$$\alpha = \int_{\gamma_o}^{\gamma_1} \frac{2 - (\sin\gamma/\sin\gamma_1)}{\sqrt{1 - (\sin\gamma/\sin\gamma_1)}} d\gamma \tag{B.11}$$

These critical values of γ_1 also correspond to the stationary values of $\Delta\omega$.